4.4 Trigonometric Functions of Any Angle

124. *Why you should learn it* *(p. 284)* A company that produces wakeboards forecasts monthly sales S over a two-year period to be

$$S = 2.7 + 0.142t + 2.2\sin\left(\frac{\pi t}{6} - \frac{\pi}{2}\right)$$

where S is measured in hundreds of units and t is the time (in months), with $t = 1$ corresponding to January 2010. Estimate sales for each month.

(a) January 2010 (b) February 2011

(c) May 2010 (d) June 2011

Introduction

What is the radius of the unit circle?

$r =$

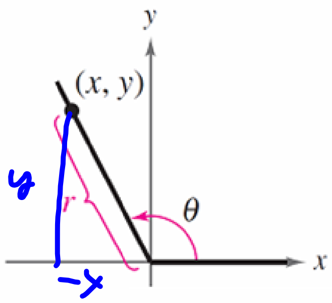
Definition of Trigonometric Functions of Any Angle

Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$.

$\sin \theta = \frac{y}{r}$ $\quad$ $\cos \theta = \frac{x}{r}$

$\tan \theta = \frac{y}{x}, \quad x \neq 0$ $\quad$ $\cot \theta = \frac{x}{y}, \quad y \neq 0$

$\sec \theta = \frac{r}{x}, \quad x \neq 0$ $\quad$ $\csc \theta = \frac{r}{y}, \quad y \neq 0$

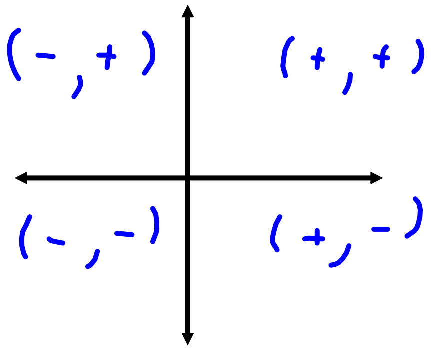


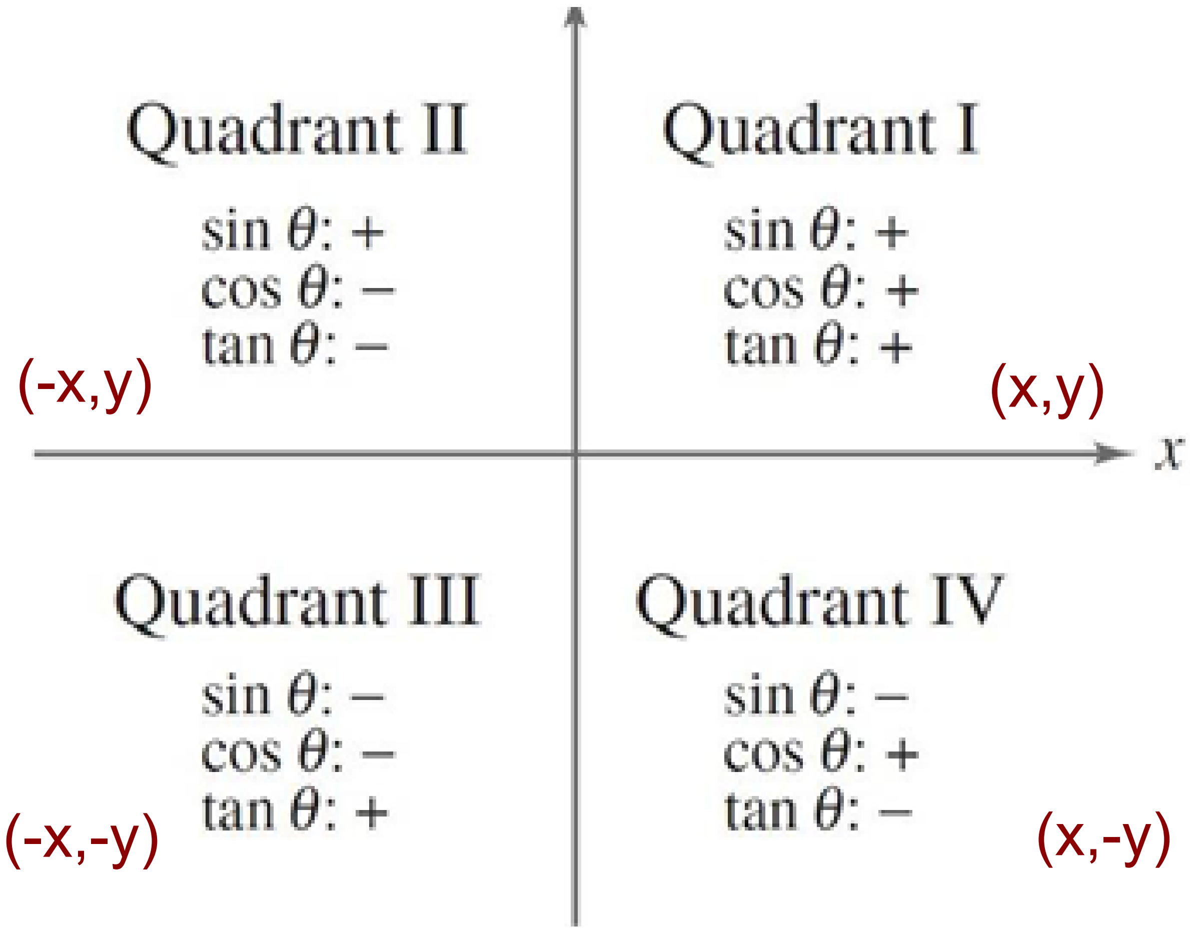
Introduction

The *signs* of the trigonometric functions in the four quadrants can be determined easily from the definitions of the functions. For instance, because

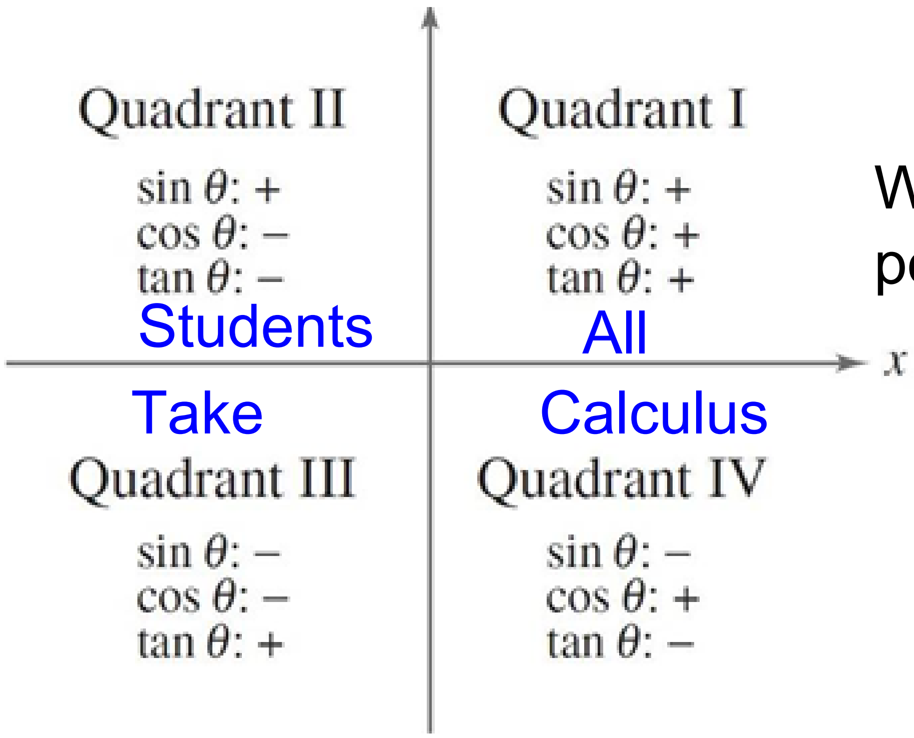
$$\cos\,\theta = \frac{x}{r}$$

it follows that cos θ is positive wherever $x > 0$, which is in Quadrants I and IV.




Quadrant II
sin θ: +
cos θ: −
tan θ: −
Quadrant I
sin θ: +
cos θ: +
tan θ: +
(-x,y)
(x,y)
x
Quadrant III
sin θ: −
cos θ: −
tan θ: +
Quadrant IV
sin θ: −
cos θ: +
tan θ: −
(-x,-y)
(x,-y)

Who is positive?



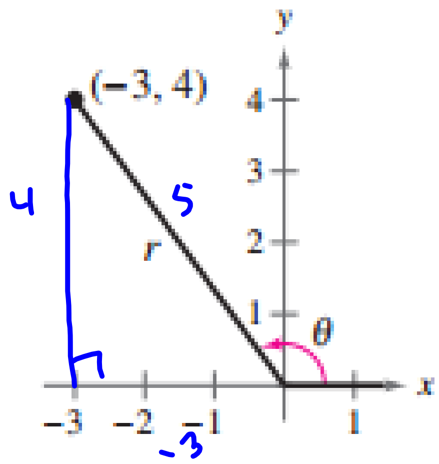
Example 1 Evaluating Trigonometric Functions

Let $(-3, 4)$ be a point on the terminal side of θ (see Figure 4.34). Find the sine, cosine, and tangent of θ.

$\sin\theta = \frac{4}{5}$

$\cos\theta = \frac{-3}{5}$

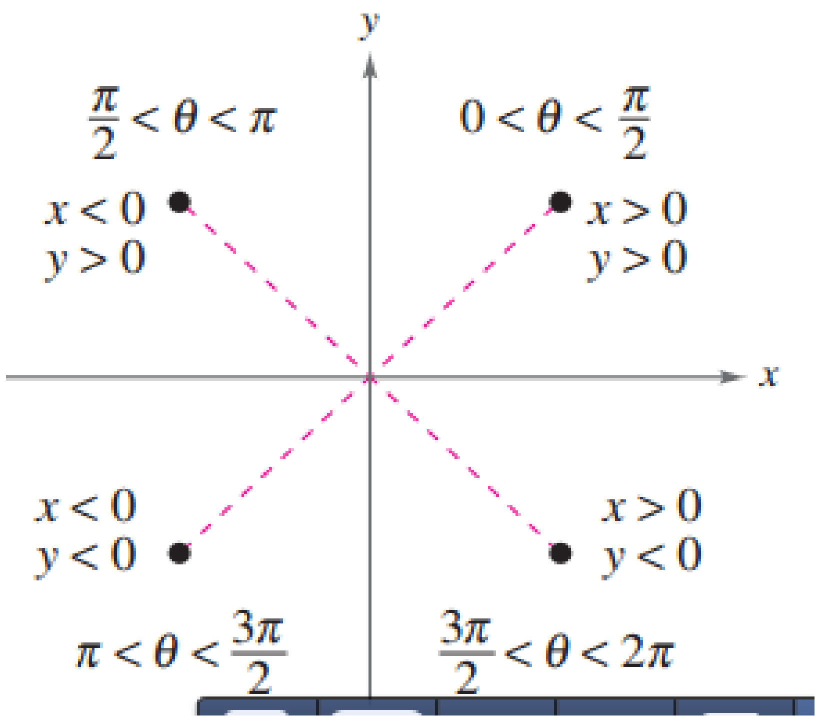
$\tan\theta = \frac{-4}{3}$



Referring to Figure 4.34, you can see that $x = -3$, $y = 4$, and

$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{(-3)^2 + 4^2}$$
$$= \sqrt{25}$$
$$= 5.$$

So, you have $\sin\theta = \frac{y}{r} = \frac{4}{5}$, $\cos\theta = \frac{x}{r} = -\frac{3}{5}$, and $\tan\theta = \frac{y}{x} = -\frac{4}{3}$.


y
$\frac{\pi}{2} < \theta < \pi$
$0 < \theta < \frac{\pi}{2}$
$x < 0$
$y > 0$
$x > 0$
$y > 0$
x
$x < 0$
$y < 0$
$x > 0$
$y < 0$
$\pi < \theta < \frac{3\pi}{2}$
$\frac{3\pi}{2} < \theta < 2\pi$

Example 2 Evaluating Trigonometric Functions

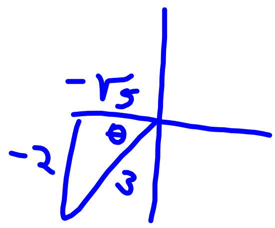
Given $\sin\theta = -\frac{2}{3}$ and $\tan\theta > 0$, find $\cos\theta$ and $\cot\theta$.

Which quadrant are we in?

Note that θ lies in Quadrant III because that is the only quadrant in which the sine is negative and the tangent is positive. Moreover, using

$\cos\theta = \frac{-\sqrt{5}}{3}$

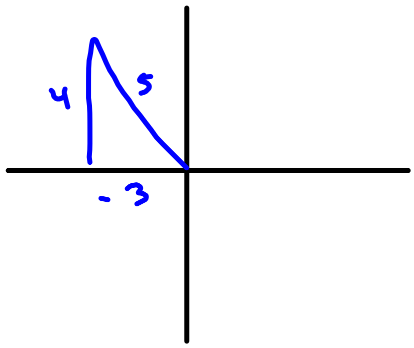
$\cot\theta = \frac{-\sqrt{5}}{-2} = \frac{\sqrt{5}}{2}$



Given $\sin\theta = \frac{4}{5}$ and $\tan\theta < 0$, find cosθ and csc θ

$\cos\theta = \frac{-3}{5}$

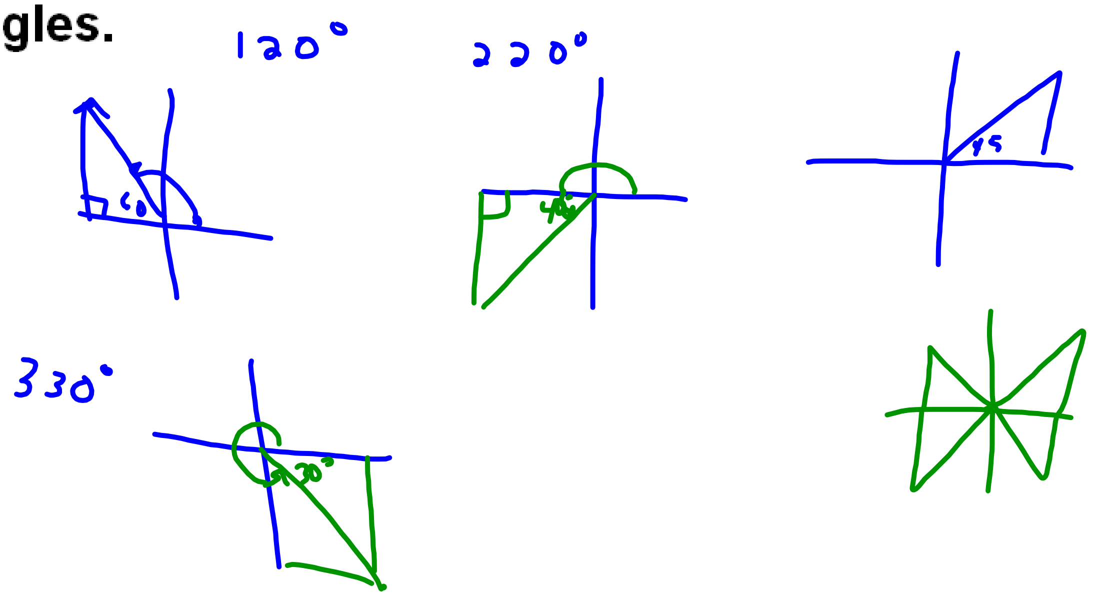
$\csc\theta = \frac{5}{4}$



Reference Angles

Reference Angles

The values of the trigonometric functions of angles greater than 90° (or less than 0°) can be determined from their values at corresponding acute angles called **reference angles.**

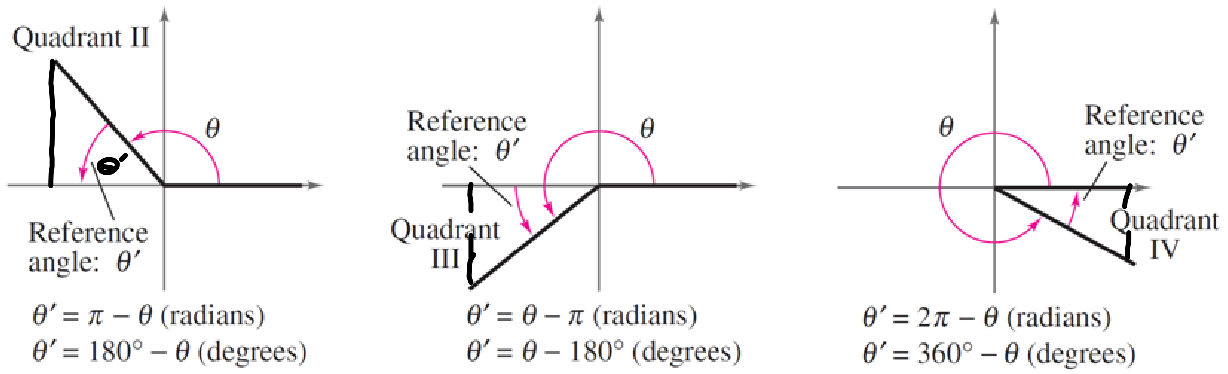


Reference Angles

Definition of Reference Angle

Let θ be an angle in standard position. Its **reference angle** is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

Reference Angles



What are the reference angles?

notice no degrees

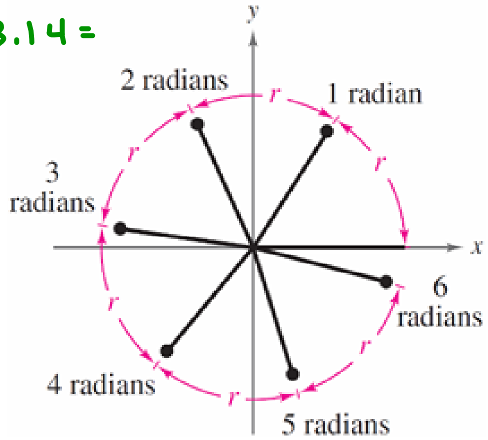
$\theta = 210^o$

-180

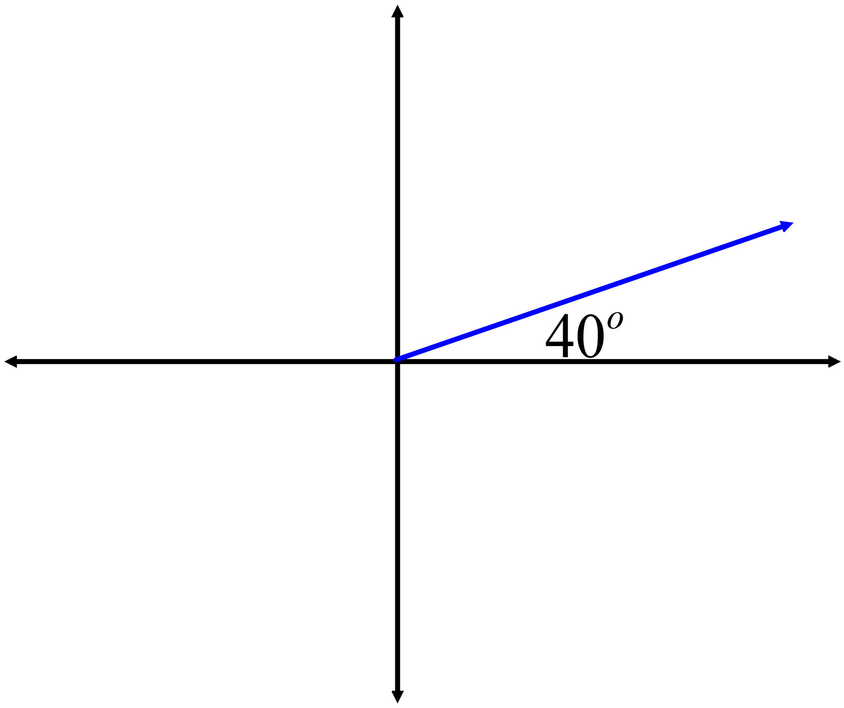
$\theta' = 30^\circ$

$\theta = 4.1$

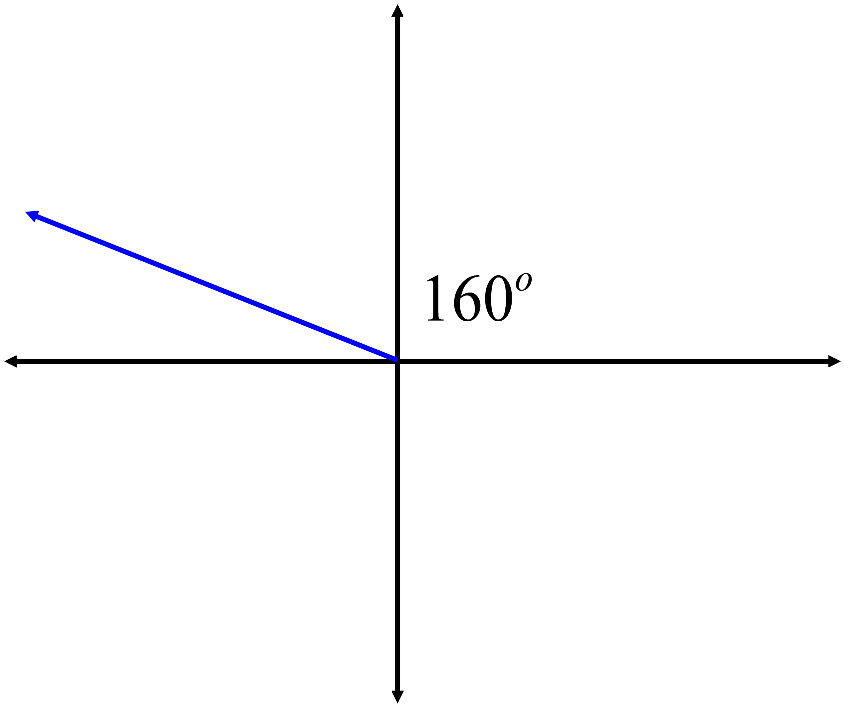
$4.1 - 3.14 =$



What is the reference angle?



What is the reference angle?



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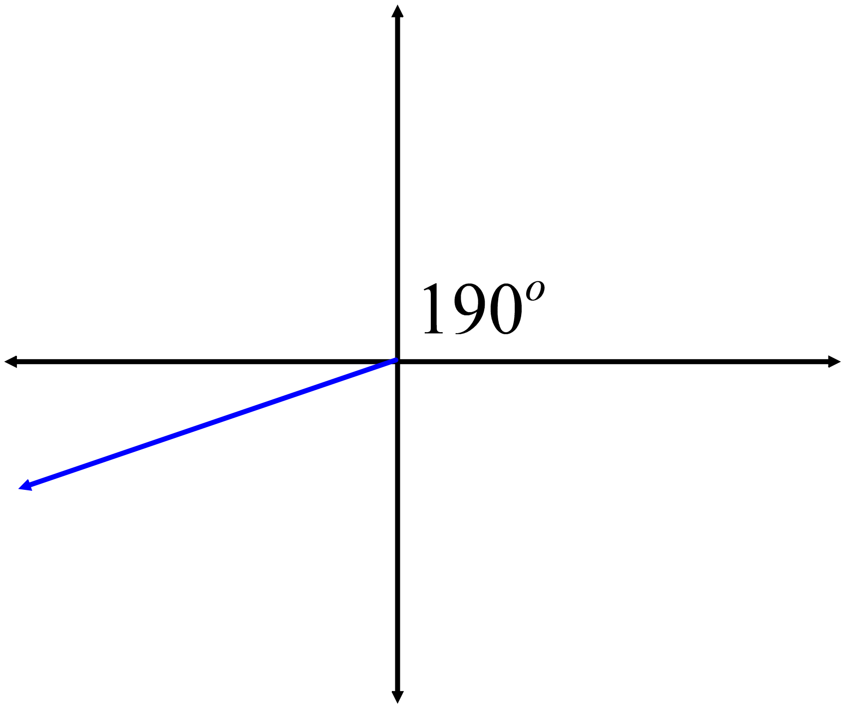
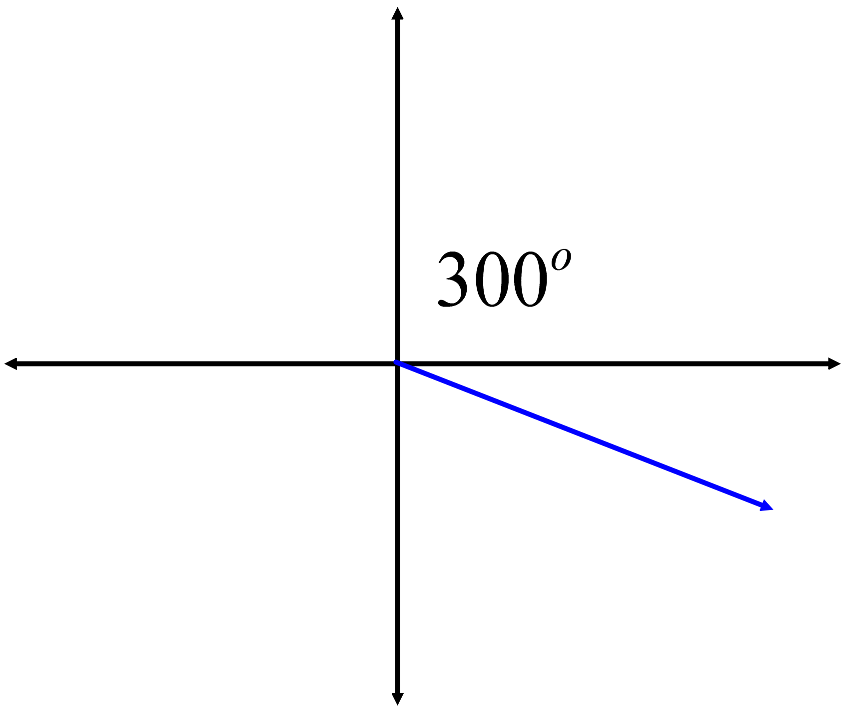
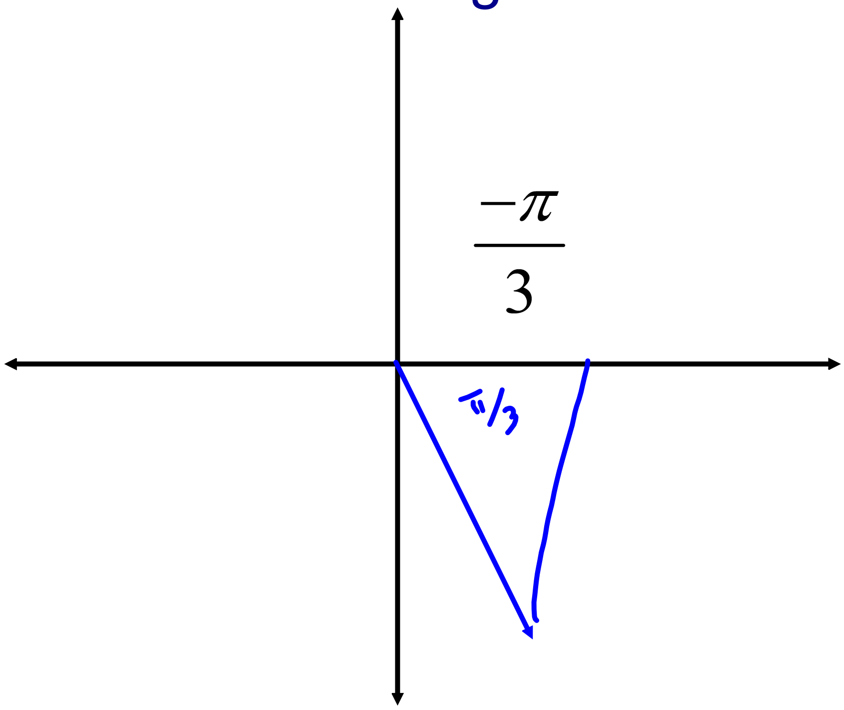


What is the reference angle?

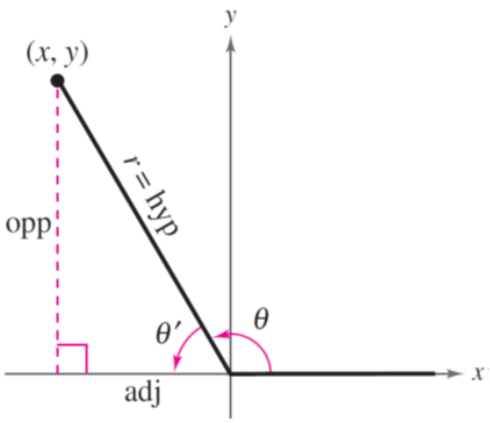


What is the reference angle?



Trigonometric Functions of Real Numbers

Trigonometric Functions of Real Numbers



Trigonometric Functions of Real Numbers

So, it follows that sin θ and sin θ' are equal, *except possibly in sign*. The same is true for tan θ and tan θ' and for the other four trigonometric functions. In all cases, the sign of the function value can be determined by the quadrant in which θ lies.

Trigonometric Functions of Real Numbers

Evaluating Trigonometric Functions of Any Angle

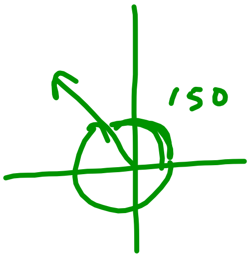
To find the value of a trigonometric function of any angle θ:

1. Determine the function value of the associated reference angle θ'.
2. Depending on the quadrant in which θ lies, affix the appropriate sign to the function value.

Trigonometric Functions of Nonacute Angles

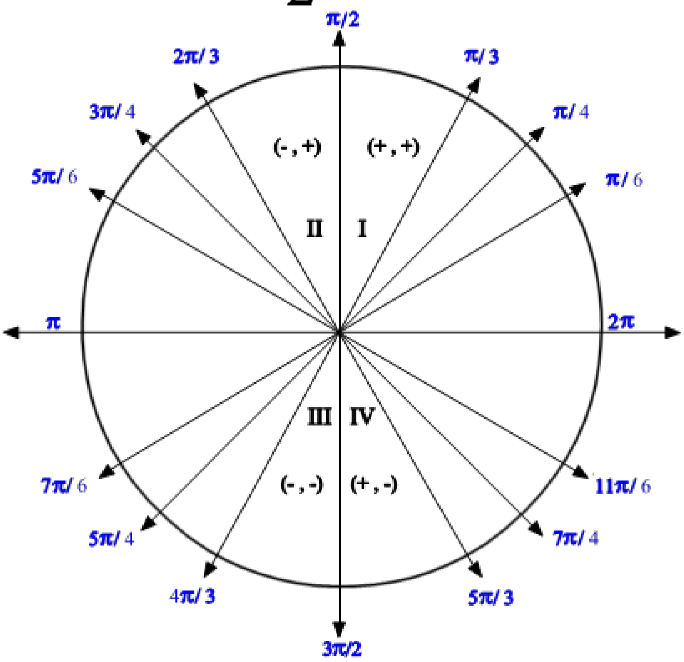
$\tan(-210^0)$ $= -\frac{\sqrt{3}}{3}$

$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

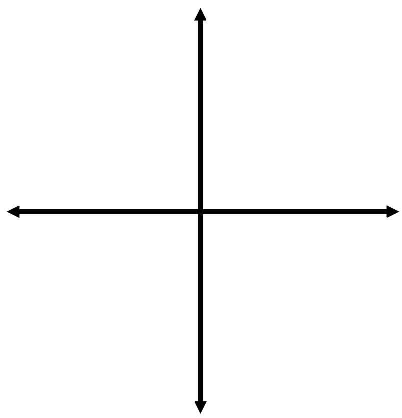


$\frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$ $\frac{1}{2} \cdot \frac{2}{\sqrt{3}}$ $-\frac{1}{\sqrt{3}}$

$$If\ \sin\theta = \frac{1}{2}\ and\ \tan\theta < 0,\ find\ \cos\theta.$$



$$If\ \tan\theta = \frac{3}{4}\ and\ \cos\theta < 0,\ find\ \sin\theta.$$

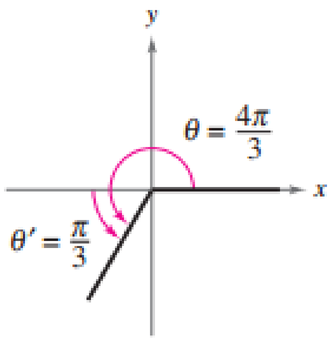


Learning the trigonometric values of common angles in the table at the left is worth the effort because doing so will increase both your efficiency and your confidence. Below is a pattern for the sine function that may help you remember the values. Reverse the order to get cosine values of the same angles.

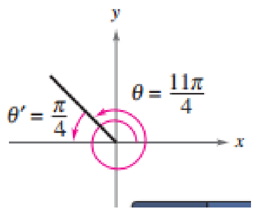
θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$

Evaluate each trigonometric function.

a. $\cos \frac{4\pi}{3}$



c. $\csc \frac{11\pi}{4}$



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Example 6 Using Trigonometric Identities

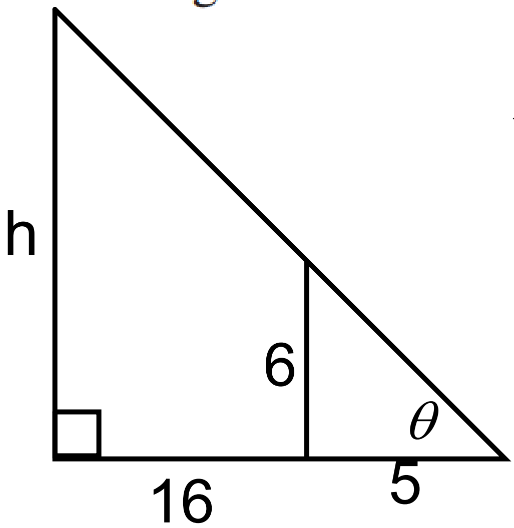
Let θ be an angle in Quadrant II such that $\sin\theta = \frac{1}{3}$. Find $\cos\theta$ by using trigonometric identities.

Solution

Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, you obtain

[redacted]

77. Geometry A six-foot person walks from the base of a streetlight directly toward the tip of the shadow cast by the streetlight. When the person is 16 feet from the streetlight and 5 feet from the tip of the streetlight's shadow, the person's shadow starts to appear beyond the streetlight's shadow.



$$\tan\theta = \frac{6}{5} \qquad \tan\theta = \frac{h}{21}$$

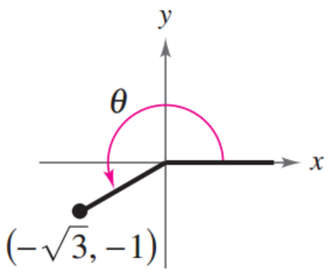
$$\frac{6}{5} = \frac{h}{21}$$

h=25.2 ft

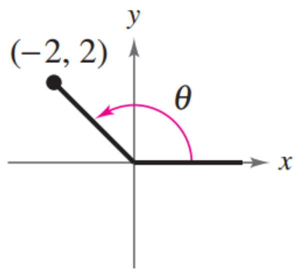
Model Homework Problems

Evaluating Trigonometric Functions **In Exercises 11–14, determine the exact values of the six trigonometric functions of the angle θ.**

13. (a)



(b)



Model Homework Problems

Evaluating Trigonometric Functions In Exercises 15–22, the point is on the terminal side of an angle in standard position. Determine the exact values of the six trigonometric functions of the angle.

19. $(-4, 10)$

Model Homework Problems

Determining a Quadrant **In Exercises 23–26, state the quadrant in which θ lies.**

26. $\tan \theta > 0$ and $\csc \theta < 0$

Model Homework Problems

Evaluating Trigonometric Functions In Exercises 27–34, find the values of the six trigonometric functions of θ.

Function Value	*Constraint*
29. $\tan\theta = -\frac{15}{8}$	$\sin\theta < 0$

Model Homework Problems

Evaluating Trigonometric Functions In Exercises 27–34, find the values of the six trigonometric functions of θ.

Function Value	*Constraint*
31. $\sec\theta = -2$	$0 \le \theta \le \pi$

Model Homework Problems

Evaluating Trigonometric Functions In Exercises 27–34, find the values of the six trigonometric functions of θ.

Function Value	*Constraint*
33. $\cot\theta$ is undefined.	$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$

Model Homework Problems

Finding a Reference Angle **In Exercises 47–58, find the reference angle θ' for the special angle θ. Sketch θ in standard position and label θ'.**

57. $\theta = \frac{11\pi}{6}$

Model Homework Problems

Finding a Reference Angle **In Exercises 59–66, find the reference angle θ'. Sketch θ in standard position and label θ'.**

61. $\theta = -292°$

Model Homework Problems

Using Trigonometric Identities **In Exercises 79–84, find the indicated trigonometric value in the specified quadrant.**

Function	*Quadrant*	*Trigonometric Value*
83. $\sec\theta = -\frac{9}{4}$	III	$\tan\theta$

Model Homework Problems

Using Trigonometric Identities **In Exercises 85–90, use the given value and the trigonometric identities to find the remaining trigonometric functions of the angle.**

87. $\tan\theta = -4,\ \cos\theta < 0$

Model Homework Problems

Solving for θ **In Exercises 103–108, find two solutions of each equation. Give your solutions in both degrees ($0° \le \theta < 360°$) and radians ($0 \le \theta < 2\pi$). Do not use a calculator.**

107. (a) $\sec \theta = -\frac{2\sqrt{3}}{3}$ (b) $\cos \theta = -\frac{1}{2}$