Warm Up

simplify

1. $\tan^2 x + 2\tan x = -1$

$\tan^2 x + 2\tan x + 1 = 0$

$(\tan x + 1)(\tan x + 1)$

solve

2. $2\cos\left(\frac{x}{2}\right) - 1 = 0$

$\cos\frac{x}{2} = \frac{1}{2}$

$2 \cdot \frac{x}{2} = 2\left(\frac{\pi}{3} + 2\pi n\right)$

$x = \frac{2\pi}{3} + 4\pi n$

$2 \cdot \frac{x}{2} = 2\left(\frac{5\pi}{3} + 2\pi n\right)$

$x = \frac{10\pi}{3} + 4\pi n$

5.4 Sum and Difference Formulas

Put the following on your notecard

You will not need to memorize these,
just know how to use them :)

Sum and Difference Formulas (See proofs on page 400.)

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$

$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

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Find the exact value of cos 75

Is 75 on the Unit Circle?

Are there 2 angles I could add together that would be on the unit circle?

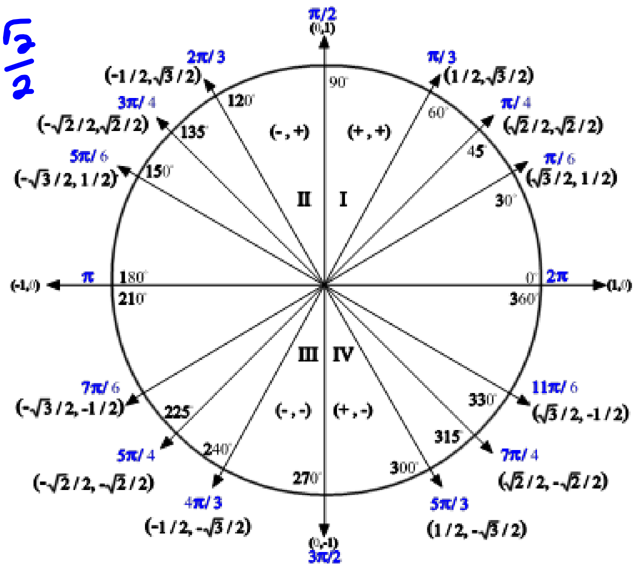
$$\cos(u + v) = \cos u \cos v - \sin u \sin v$$

$$\cos(30 + 45) = \cos 30 \cos 45 - \sin 30 \sin 45$$

$$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$$

$$\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

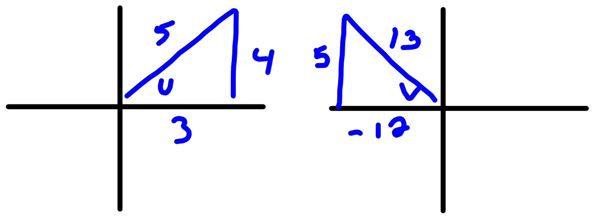
$$\frac{\sqrt{6} - \sqrt{2}}{4}$$

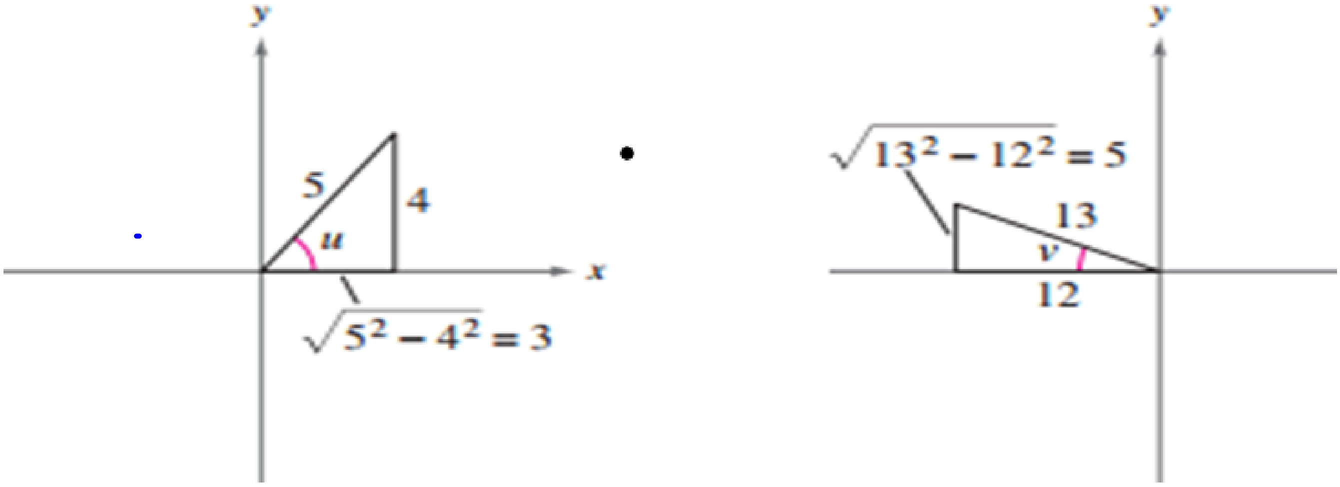


Find the exact value of $\sin(u + v)$ given

$$\sin u = \frac{4}{5}, \text{ where } 0 < u < \frac{\pi}{2} \quad \text{and} \quad \cos v = -\frac{12}{13}, \text{ where } \frac{\pi}{2} < v < \pi.$$

Start by making triangles for what you know.





Solution

Because $\sin u = 4/5$ and u is in Quadrant I, $\cos u = 3/5$, as shown in Figure 5.19. Because $\cos v = -12/13$ and v is in Quadrant II, $\sin v = 5/13$, as shown in Figure 5.20. You can find $\sin(u + v)$ as follows.

$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

$$= \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) = -\frac{48}{65} + \frac{15}{65} = -\frac{33}{65}$$

Find the exact value for $\cos\frac{\pi}{12}$

$$\frac{\pi}{12}=\frac{\pi}{3}-\frac{\pi}{4}$$

$$\cos(u-v)=\cos u\cos v+\sin u\sin v$$

$$\cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$\frac{1}{2}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}$$

$$\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}=\frac{\sqrt{2}+\sqrt{6}}{4}$$

Write $\cos(\arctan 1 + \arccos x)$ as an algebraic expression.

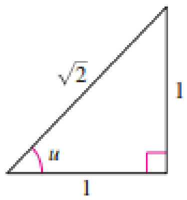


Figure 5.21

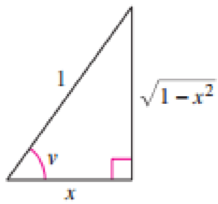


Figure 5.22

Solution

This expression fits the formula for $\cos(u + v)$. Angles

$u = \arctan 1$ and $v = \arccos x$

are shown in Figures 5.21 and 5.22, respectively.

$$\cos(u + v) = \cos(\arctan 1)\cos(\arccos x) - \sin(\arctan 1)\sin(\arccos x)$$

$$= \frac{1}{\sqrt{2}} \cdot x - \frac{1}{\sqrt{2}} \cdot \sqrt{1 - x^2}$$

$$= \frac{x - \sqrt{1 - x^2}}{\sqrt{2}}.$$

Prove the cofunction identity $\cos\left(\frac{\pi}{2} - x\right) = \sin x$.

Solution

Using the formula for $\cos(u - v)$, you have

$$\begin{aligned}\cos\left(\frac{\pi}{2} - x\right) &= \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x \\ &= (0)(\cos x) + (1)(\sin x) \\ &= \sin x.\end{aligned}$$

Simplify each expression.

a. $\cos\left(\theta - \frac{3\pi}{2}\right)$

b. $\tan(\theta + 3\pi)$

Solution

a. Using the formula for $\cos(u - v)$, you have

$$\begin{aligned}\cos\left(\theta - \frac{3\pi}{2}\right) &= \cos\theta\cos\frac{3\pi}{2} + \sin\theta\sin\frac{3\pi}{2}\\ &= (\cos\theta)(0) + (\sin\theta)(-1)\\ &= -\sin\theta.\end{aligned}$$

b. Using the formula for $\tan(u + v)$, you have

$$\begin{aligned}\tan(\theta + 3\pi) &= \frac{\tan\theta + \tan 3\pi}{1 - \tan\theta\tan 3\pi}\\ &= \frac{\tan\theta + 0}{1 - (\tan\theta)(0)}\\ &= \tan\theta.\end{aligned}$$

Note that the period of $\tan\theta$ is π, so the period of $\tan(\theta + 3\pi)$ is the same as the period of $\tan\theta$.

Find all solutions of

$$\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right) = -1$$

in the interval $[0, 2\pi)$.

Algebraic Solution

Using sum and difference formulas, rewrite the equation as

$$\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4} + \sin x \cos \frac{\pi}{4} - \cos x \sin \frac{\pi}{4} = -1$$

$$2 \sin x \cos \frac{\pi}{4} = -1$$

$$2(\sin x)\left(\frac{\sqrt{2}}{2}\right) = -1$$

$$\sin x = -\frac{1}{\sqrt{2}}$$

$$\sin x = -\frac{\sqrt{2}}{2}.$$

So, the only solutions in the interval $[0, 2\pi)$ are

$$x = \frac{5\pi}{4} \quad \text{and} \quad x = \frac{7\pi}{4}.$$

Graphical Solution

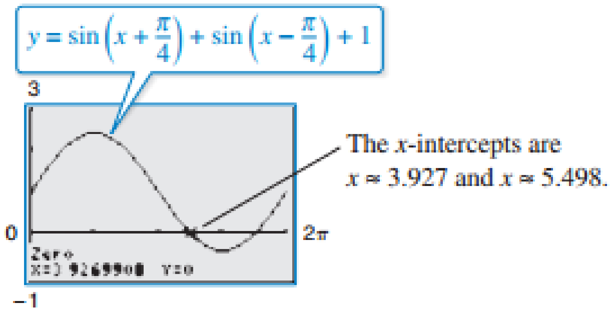


Figure 5.23

From Figure 5.23, you can conclude that the approximate solutions in the interval $[0, 2\pi)$ are

$$x \approx 3.927 \approx \frac{5\pi}{4} \text{ and } x \approx 5.498 \approx \frac{7\pi}{4}.$$

Example 7 An Application from Calculus

Verify that

$$\frac{\cos(x+h)-\cos x}{h} = (\cos x)\left(\frac{\cos h - 1}{h}\right) - (\sin x)\left(\frac{\sin h}{h}\right)$$

where $h \neq 0$.

Solution

Using the formula for $\cos(u+v)$, you have

$$\frac{\cos(x+h)-\cos x}{h} = \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$

$$= \frac{\cos x(\cos h - 1)}{h} - \frac{\sin x \sin h}{h}$$

$$= (\cos x)\left(\frac{\cos h - 1}{h}\right) - (\sin x)\left(\frac{\sin h}{h}\right).$$

Answer these with a partner on a paper to hand in

Use a sum or difference formula to find the exact value of:

$\tan 255^{o}$ (300-45)

$\sin 285^{o}$ (315-30)

Find the exact value of the trigonometric function given that $\sin u = \frac{4}{5}$ and $\cos v = \frac{-7}{25}$.

Both u and v are in Quadrant II.

sin (u + v)

tan (u-v)

cos (u+ v)

sin (u-v)

cos (u-v)

tan (u+v)

Use sum / difference formulas to prove the cofunction identity:

$$\sin\left(x-\frac{3\pi}{2}\right)=\cos x$$

Find all solutions in the interval [0, 2π)

$$\sin\left(x+\frac{\pi}{2}\right)-\sin\left(x-\frac{\pi}{2}\right)=\sqrt{2}$$

Verify

$$\sin\left(\pi - x\right) = \sin x$$

How do you simplify expressions and solve equations that contain sums of differences of angles?