5.5 Multiple-Angle and Product-to-Sum Formulas

Double Angle Formulas

Double-Angle Formulas (See the proofs on page 401.)

$$\sin 2u = 2 \sin u \cos u$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

$$\begin{aligned} \cos 2u &= \cos^2 u - \sin^2 u \\ &= 2\cos^2 u - 1 \\ &= 1 - 2\sin^2 u \end{aligned}$$

Solve 2cosx + sin2x=0

$\sin 2u = 2 \sin u \cos u$

$2\cos x + 2\sin x \cos x = 0$

$2\cos x(1 + \sin x) = 0$

$2\cos x = 0$ $\quad$ $1 + \sin x = 0$

$\cos x = 0$ $\quad$ $\sin x = -1$

Solution

Begin by rewriting the equation so that it involves functions of x (rather than $2x$). Then factor and solve as usual.

$$2\cos x + \sin 2x = 0$$ Write original equation.

$$2\cos x + 2\sin x\cos x = 0$$ Double-angle formula

$$2\cos x(1 + \sin x) = 0$$ Factor.

$$2\cos x = 0 \qquad 1 + \sin x = 0$$ Set factors equal to zero.

$$\cos x = 0 \qquad \sin x = -1$$ Isolate trigonometric functions.

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = \frac{3\pi}{2}$$ Solutions in $[0, 2\pi)$

So, the general solution is

$$x = \frac{\pi}{2} + 2n\pi \quad \text{and} \quad x = \frac{3\pi}{2} + 2n\pi$$ General solution

where n is an integer. Try verifying this solution graphically.

Partner: One paper per team

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Given: $\cos\theta = \frac{5}{13}$ $\frac{3\pi}{2} < \theta < 2\pi$

Find:

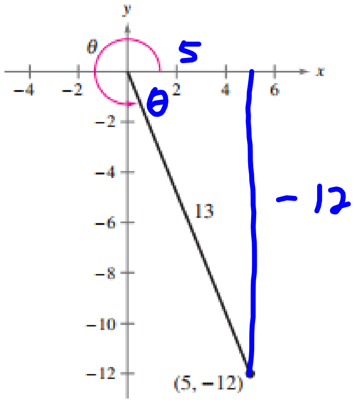
cos 2θ

sin 2θ

tan 2θ

Use the following to find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

$$\cos\theta = \frac{5}{13}, \qquad \frac{3\pi}{2} < \theta < 2\pi$$



In Figure 5.24, you can see that

$$\sin\theta = \frac{y}{r} = -\frac{12}{13}$$

and

$$\tan\theta = -\frac{12}{5}.$$

Consequently, using each of the double-angle formulas, you can write the double angles as follows.

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(-\frac{12}{13}\right)\left(\frac{5}{13}\right) = -\frac{120}{169}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{25}{169}\right) - 1 = -\frac{119}{169}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2(-12/5)}{1-(-12/5)^2} = \frac{120}{119}$$

The double-angle formulas are not restricted to the angles 2θ and θ. Other *double* combinations, such as 4θ and 2θ or 6θ and 3θ, are also valid. Here are two examples.

$$\sin 4\theta = 2 \sin 2\theta \cos 2\theta \qquad \text{and} \qquad \cos 6\theta = \cos^2 3\theta - \sin^2 3\theta$$

By using double-angle formulas together with the sum formulas derived in the preceding section, you can form other multiple-angle formulas.

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Power Reducing Formulas

Power-Reducing Formulas (See the proofs on page 401.)

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

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Half-Angle Formulas

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

The signs of $\sin\frac{u}{2}$ and $\cos\frac{u}{2}$ depend on the quadrant in which $\frac{u}{2}$ lies.

Using the half angle formula

Find the exact value of sin 105°.

What angle would we use to get 105, that is on the unit circle? What if I doubled 105 to get 210, is that on the unit circle? Now use the half angle formula.

Solution

Begin by noting that 105° is half of 210°. Then, using the half-angle formula for $\sin(u/2)$ and the fact that 105° lies in Quadrant II, you have

$$\sin 105° = \sqrt{\frac{1 - \cos 210°}{2}}$$

$$= \sqrt{\frac{1 - (-\cos 30°)}{2}}$$

$$= \sqrt{\frac{1 + (\sqrt{3}/2)}{2}} = \frac{\sqrt{2 + \sqrt{3}}}{2}.$$

The positive square root is chosen because $\sin \theta$ is positive in Quadrant II.

Technology Tip

Use your calculator to verify the result obtained in Example 5. That is, evaluate $\sin 105°$ and $\left(\sqrt{2+\sqrt{3}}\right)/2$. You will notice that both expressions yield the same result.

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Product-to-Sum Formulas

$$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$$

$$\sin u \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \frac{1}{2}[\sin(u + v) - \sin(u - v)]$$

Rewrite the product as a sum or difference.

$\cos 5x \sin 4x$

$$\cos u \sin v = \frac{1}{2}[\sin(u + v) - \sin(u - v)]$$

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Solution

Using the appropriate product-to-sum formula, you obtain

$$\cos 5x \sin 4x = \frac{1}{2}[\sin(5x + 4x) - \sin(5x - 4x)]$$

$$= \frac{1}{2}\sin 9x - \frac{1}{2}\sin x.$$

Can not simplify any more

Find all solutions of $1 + \cos^2 x = 2\cos^2 \frac{x}{2}$ in the interval $[0, 2\pi)$.

Algebraic Solution

$$1 + \cos^2 x = 2\cos^2 \frac{x}{2}$$ Write original equation.

$$1 + \cos^2 x = 2\left(\pm\sqrt{\frac{1 + \cos x}{2}}\right)^2$$ Half-angle formula

$$1 + \cos^2 x = 1 + \cos x$$ Simplify.

$$\cos^2 x - \cos x = 0$$ Simplify.

$$\cos x(\cos x - 1) = 0$$ Factor.

By setting the factors $\cos x$ and $\cos x - 1$ equal to zero, you find that the solutions in the interval $[0, 2\pi)$ are $x = \pi/2$, $x = 3\pi/2$, and $x = 0$.

Graphical Solution

The x-intercepts are $x = 0$, $x \approx 1.571$, and $x \approx 4.712$.

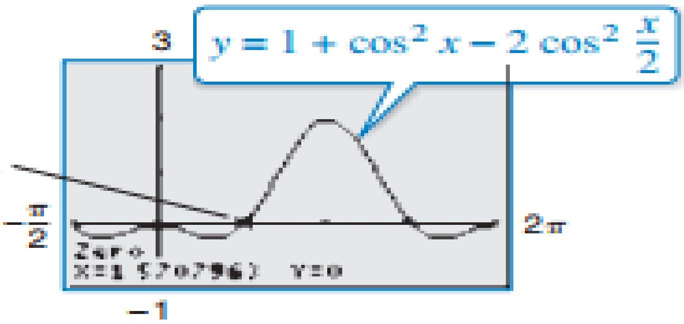

Figure 5.26

From Figure 5.26, you can conclude that the approximate solutions of $1 + \cos^2 x = 2\cos^2 \frac{x}{2}$ in the interval $[0, 2\pi)$

are $x = 0$, $x \approx 1.571 \approx$

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Sum-to-Product Formulas (See proof on page 402.)

$$\sin u + \sin v = 2 \sin\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right)$$

$$\sin u - \sin v = 2 \cos\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right)$$

$$\cos u + \cos v = 2 \cos\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right)$$

$$\cos u - \cos v = -2 \sin\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right)$$

Find the exact value of $\cos 195^o + \cos 105^o$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

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Solution

Using the appropriate sum-to-product formula, you obtain

$$\begin{aligned}\cos 195° + \cos 105° &= 2\cos\left(\frac{195° + 105°}{2}\right)\cos\left(\frac{195° - 105°}{2}\right)\\ &= 2\cos 150° \cos 45°\\ &= 2\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= -\frac{\sqrt{6}}{2}.\end{aligned}$$

Technology Tip

You can use a graphing utility to verify the solution in Example 7. Graph $y_1 = \cos 5x \sin 4x$ and $y_2 = \frac{1}{2}\sin 9x - \frac{1}{2}\sin x$ in the same viewing window. Notice that the graphs coincide. So, you can conclude that the two expressions are equivalent.

Find all solutions of

$$\sin 5x + \sin 3x = 0$$

in the interval $[0, 2\pi)$.

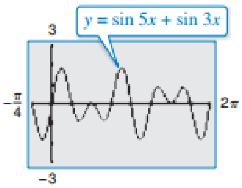


Figure 5.27

Notice that the general solution is

$$x = \frac{n\pi}{4}$$

where n is an integer.

Solution

$$\sin 5x + \sin 3x = 0 \quad \text{Write original equation.}$$

$$2\sin\left(\frac{5x + 3x}{2}\right)\cos\left(\frac{5x - 3x}{2}\right) = 0 \quad \text{Sum-to-product formula}$$

$$2\sin 4x \cos x = 0 \quad \text{Simplify.}$$

By setting the factor $\sin 4x$ equal to zero, you can find that the solutions in the interval $[0, 2\pi)$ are

$$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}.$$

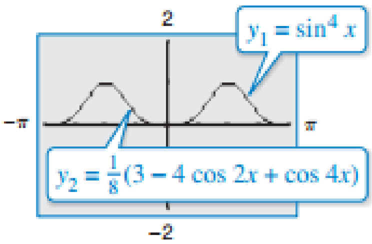
The equation $\cos x = 0$ yields no additional solutions. You can use a graphing utility to confirm the solutions, as shown in Figure 5.27.

Example 4 Reducing a Power

Rewrite $\sin^4 x$ as a sum of first powers of the cosines of multiple angles.

$$
\begin{aligned}
\sin^4 x &= (\sin^2 x)^2 && \text{Property of exponents} \\
&= \left(\frac{1 - \cos 2x}{2}\right)^2 && \text{Power-reducing formula} \\
&= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x) && \text{Expand binomial.} \\
&= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1 + \cos 4x}{2}\right) && \text{Power-reducing formula} \\
&= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x && \text{Distributive Property} \\
&= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x && \text{Simplify.} \\
&= \frac{1}{8}(3 - 4\cos 2x + \cos 4x) && \text{Factor.}
\end{aligned}
$$

You can use a graphing utility to check this result, as shown in Figure 5.25. Notice that the graphs coincide.



5.5 hw is long, and at times challenging!
You have all of the background knowledge, skills, and formulas necessary to complete the problems.
I don't want to beat you to death with example problems, so....

- think
- apply
- attack
- be creative
- clearly show all work
- take your time

Rewrite **$\sin^4 x$** as first powers

Given: $\cos\theta = \frac{7}{25}$, $\frac{3\pi}{2} < \theta < 2\pi$

Find: $\cos\frac{\theta}{2}$

Use half angle formulas to find: **sin 105°**

Write **cos 3x cos 2x** as a sum or difference

Write **cos 4x + cos 2x** as a product

Solve: $2 \cos x + \sin 2x = 0$