6.3 Vectors

http://www.youtube.com/watch?v=A05n32Bl0aY

Component Form of a Vector

A vector whose initial point is at the origin (0, 0) can be represented by the coordinates of its terminal point (v_1, v_2).

$$\mathbf{v} = < v_1 , v_2 >$$

The component form of the vector with initial point $P = (x_1, y_1)$ and terminal point $Q = (x_2, y_2)$ is:

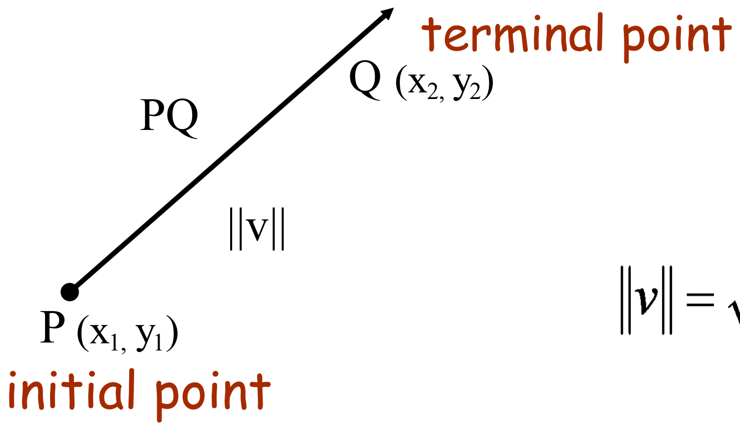
$$< v_1, v_2 >= \left(x_2 - x_1, y_2 - y_1\right)$$

The **magnitude** (or length) of **v** is:

$$\|v\| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

$$\|v\| = \sqrt{v_1^{\ 2} + v_2^{\ 2}}$$

Vector: has magnitude (length) and direction



magnitude

$$\|v\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Component Form of a Vector

The component form of the vector with initial point $P(p_1, p_2)$ and terminal point $Q(q_1, q_2)$ is given by

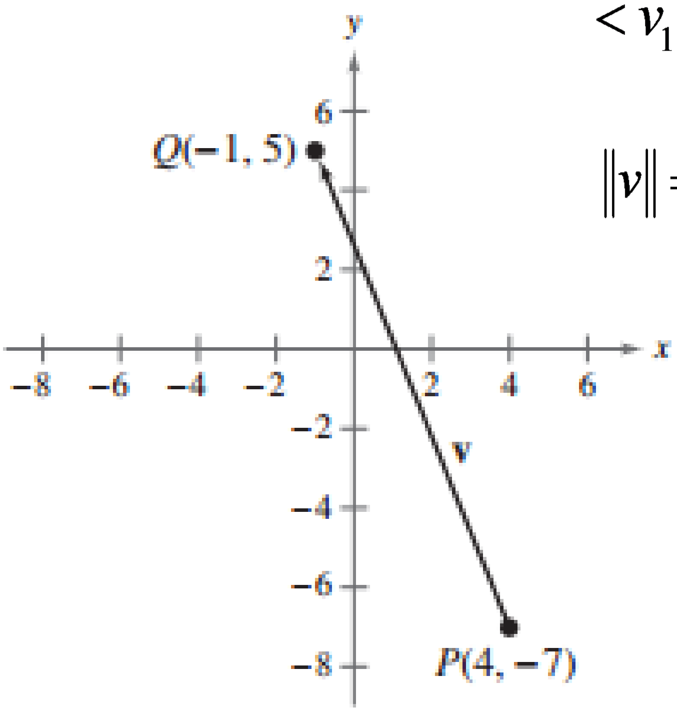
$$\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle = \mathbf{v}.$$

The **magnitude** (or length) of $\mathbf{v}$ is given by

$$\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}.$$

If $\|\mathbf{v}\| = 1$, then $\mathbf{v}$ is a **unit vector.** Moreover, $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v}$ is the zero vector $\mathbf{0}$.

Find the component form and magnitude of the vector **v** that has initial point (4, −7) and terminal point (−1, 5).



$$< v_1, v_2 >= \left(x_2 - x_1, y_2 - y_1\right)$$

$$\|v\| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

$$\|v\| = \sqrt{(-1-4)^2 + (5 - -7)^2}$$

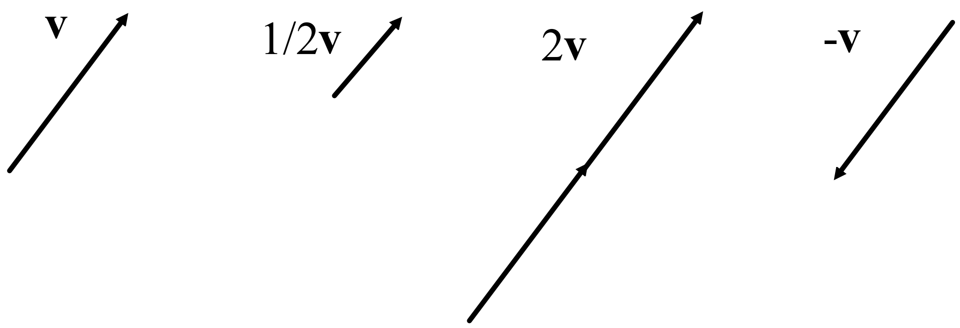
$$\sqrt{25 + 144}$$

$$\sqrt{169}$$

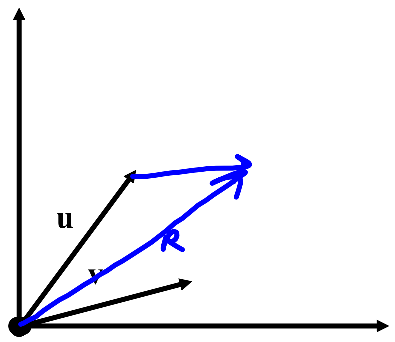
$$\|v\| = 13$$

Vector Operations

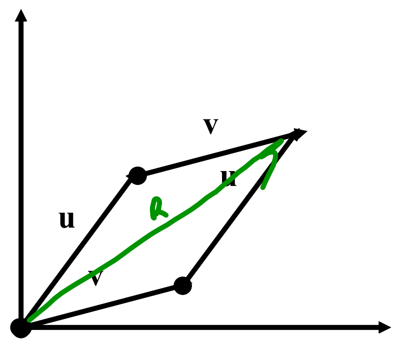
Geometrically, the product of a vector **v** and a scalar *k* is . . .
the vector that is k times as long as **v**
If *k* is positive, *k***v** has the __same__ direction as **v**, and if *k* is negative, *k***v** has the __opposite__ direction.

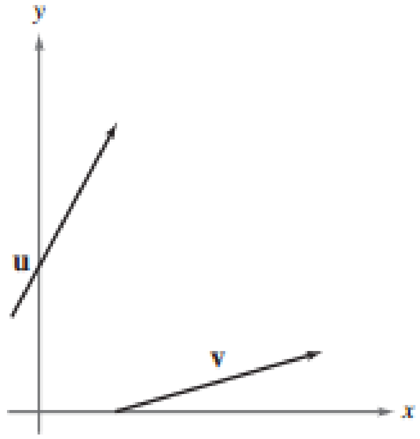


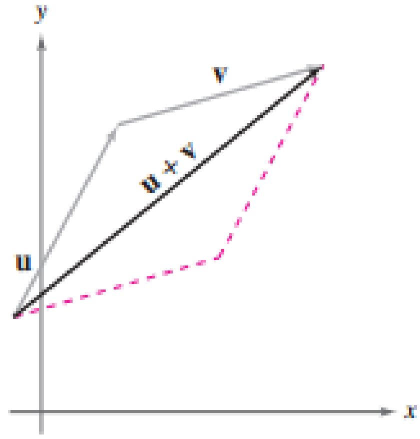
To add two vectors geometrically, position them so the initial point of one coincides with the terminal point of the other

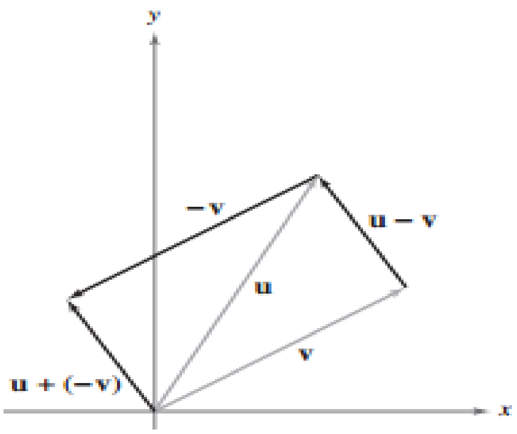


This technique is called the __parallelogram law__ for vector addition because the vector **u** + **v**, often called the _resultant_ of vector addition, is . . . the diagonal of a parallelogram with **u** and **v** as sides




y
u
v
x


y
v
u + v
u
x


y
−v
u − v
u
v
u + (−v)
x

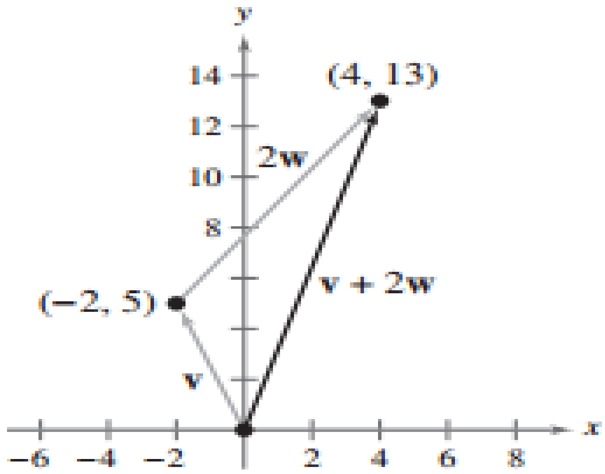
Definition of Vector Addition and Scalar Multiplication

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and let k be a scalar (a real number). Then the **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle \quad \text{Sum}$$

and the **scalar multiple** of k times $\mathbf{u}$ is the vector

$$k\mathbf{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle. \quad \text{Scalar multiple}$$

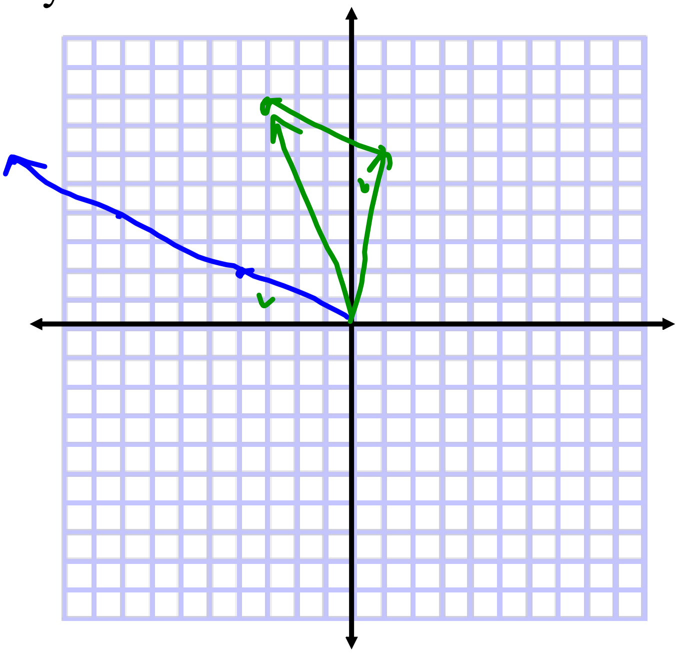

y
14
12
10
8
(4, 13)
2w
v + 2w
(−2, 5)
v
x
−6
−4
−2
2
4
6
8

Properties of Vector Addition and Scalar Multiplication

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors and let c and d be scalars. Then the following properties are true.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $c(d\mathbf{u}) = (cd)\mathbf{u}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $1(\mathbf{u}) = \mathbf{u}$, $0(\mathbf{u}) = \mathbf{0}$
9. $\|c\mathbf{v}\| = |c| \, \|\mathbf{v}\|$

Let **u** = ⟨1, 6⟩ and **v** = ⟨– 4, 2⟩.
Sketch the operations geometrically.
Then find:



(a) 3**v**

3⟨−4, 2⟩

⟨−12, 6⟩

(b) u + **v**

⟨1, 6⟩ + ⟨−4, 2⟩

⟨−3, 8⟩

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and let k be a scalar

$\mathbf{u} + \mathbf{v} = < u_1 + v_1, u_2 + v_2 >$ *(vector addition)*

$k\mathbf{u} = < ku_1 , ku_2 >$ *(Scalar multiplication)*

u = <4,6>   v = <-2, 5>

Find the following: ~~Graphically~~ and algebraically

a. 2u - 3v

b. u + 2v

c. 2u + 3v

Component form
$\langle 2, 4 \rangle$

Vector Activity

Linear combination
Form

$2i + 4j$

http://www.youtube.com/watch?v=fVq4_HhBK8Y

Airplane

u = unit vector = $\frac{v}{\|v\|} = \left(\frac{1}{\|v\|}\right)v$

$$\|v\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Find a unit vector in the direction of $\mathbf{v} = \langle -8, 6 \rangle$.

$$\frac{v}{\|v\|}$$

Find a unit vector u in the direction of $v = \langle 7, -3 \rangle$
and verify that the result has magnitude 1.

find the unit vector
then use in

$$\|v\| = \sqrt{(v_1)^2 + (v_2)^2}$$

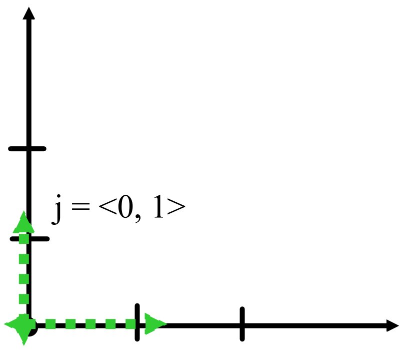
Find the vector v with the given magnitude and same direction as u.

$\|v\| = 3 \qquad u = \langle 4, -4 \rangle$

find the unit vector
multiply it by the magnitude

Standard Unit Vectors:

i = <1, 0>   j = <0, 1>



Linear Combination

Vector $\mathbf{v} = \langle v_1, v_2 \rangle$

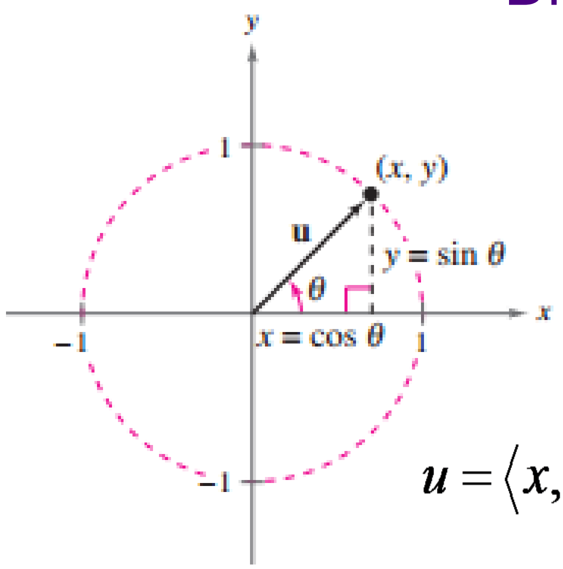
can also be represented as:

$$\begin{aligned}\mathbf{v} &= \langle v_1, v_2 \rangle \\ &= v_1\langle 1, 0 \rangle + v_2\langle 0, 1 \rangle \\ &= v_1\mathbf{i} + v_2\mathbf{j}\end{aligned}$$

Let $u = -3i+8j$ and $v = 2i - j$.
Find $2u - 3v$

Let $\mathbf{v} = \langle -5, 3 \rangle$. Write $\mathbf{v}$ as a linear combination of the standard unit vectors $\mathbf{i}$ and $\mathbf{j}$.

Direction Angles



If **u** is a unit vector such that θ is the angle (measured counter clockwise) from the positive x-axis to u, then the terminal point of **u** lies on the unit circle

$$u = \langle x, y \rangle = \langle \cos\theta, \sin\theta \rangle = \cos\theta i + \sin\theta j$$

The angle θ is the direction angle of vector u.

$$v = \|v\|\langle \cos\theta, \sin\theta \rangle = \|v\|(\cos\theta)i + \|v\|(\sin\theta)j$$

Vector **v** has direction angle $\theta = 30^o$ and magnitude 6. Find **v**.

$$v = \|6\|\left\langle \cos 30^o, \sin 30^o \right\rangle = \|v\|(\cos\theta)i + \|v\|(\sin\theta)j$$

v=ai + bj

$$\tan\theta = \frac{\|v\|\sin\theta}{\|v\|\cos\theta} = \frac{b}{a}$$

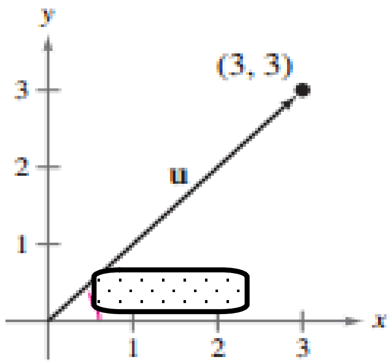
You must pay attention to what quadrant you are in!

Find the direction angle of each vector.

a. u = 3i + 3j

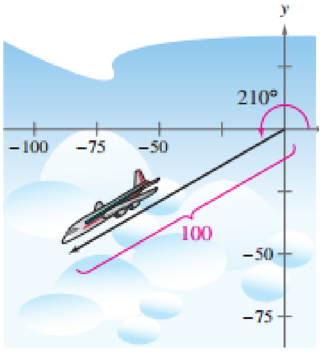
$$\tan\theta = \frac{\|v\|\sin\theta}{\|v\|\cos\theta} = \frac{b}{a}$$

u=ai+bj



b. v = 3i - 4j

Find the component form of the vector that represents the velocity of an airplane descending at a speed of 100 miles per hour at an angle of 30° below the horizontal, as shown in Figure 6.32.



The velocity vector **v** has a magnitude of 100 and a direction angle of $\theta = 210°$

$$\mathbf{v} = \|\mathbf{v}\|(\cos\theta)\mathbf{i} + \|\mathbf{v}\|(\sin\theta)\mathbf{j}$$
$$= 100(\cos 210°)\mathbf{i} + 100(\sin 210°)\mathbf{j}$$
$$= 100\left(-\frac{\sqrt{3}}{2}\right)\mathbf{i} + 100\left(-\frac{1}{2}\right)\mathbf{j}$$
$$= -50\sqrt{3}\,\mathbf{i} - 50\mathbf{j}$$
$$= \langle -50\sqrt{3}, -50 \rangle$$

To find the angle:

$$\tan\theta = \frac{-50}{-50\sqrt{3}}$$

You can check that **v** has a magnitude of 100 as follows.

$$\|\mathbf{v}\| = \sqrt{(-50\sqrt{3})^2 + (-50)^2}$$
$$= \sqrt{7500 + 2500}$$
$$= \sqrt{10{,}000}$$
$$= 100$$

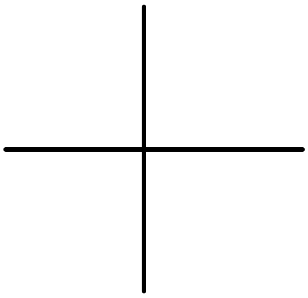
Solution checks. ✓

Using Vectors to Find Speed and Direction *Example 10 p.416*

An airplane is traveling at a speed of 500 mph with a bearing of 330°. The airplane encounters a wind blowing 70 mph in the direction N 45° E. What are the resultant speed and direction of the airplane? Bearing starts at N.

What is the magnitude?

Plane:



Wind:

Plane + Wind

magnitude: $\|v\| =$

$\|v\| =$

direction:

$\tan\theta =$

$\theta =$

The initial point of a vector is (-6, 4) and the terminal point is (0, 1)
Write a linear combination of the standard vector.

Find the component form of V given its magnitude and angle.

$||v|| = 4$, $\theta = 45^{o}$

Applications of Vectors

force / velocity / tension = magnitude / length

also think: Geometry, right triangle trig, law of cosines and sines

Highlight of Important Vocab / Formulas:

Component Form: $\mathbf{v} = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$
found by subtracting x values and y values of points

Magnitude: $||v||$
found by: $\sqrt{v_1^2 + v_2^2}$ *or the distance formula*

Unit Vector: $\dfrac{\mathbf{v}}{||\mathbf{v}||}$

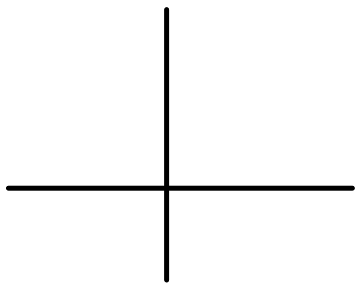
Linear Combination of v: $\mathbf{v}_1\mathbf{i} + \mathbf{v}_2\mathbf{j}$

Direction Angles: $\mathbf{v} = ||\mathbf{v}|| \cos\theta\ \mathbf{i} + ||\mathbf{v}|| \sin\theta\ \mathbf{j}$

If v = ai + bj, then $\tan\theta = \dfrac{b}{a}$

Using Vectors to Find Speed and Direction

An airplane is traveling at a speed of 500 mph with a bearing of 330°. The airplane encounters a wind blowing 70 mph in the direction N 45° E. What are the resultant speed and direction of the airplane?

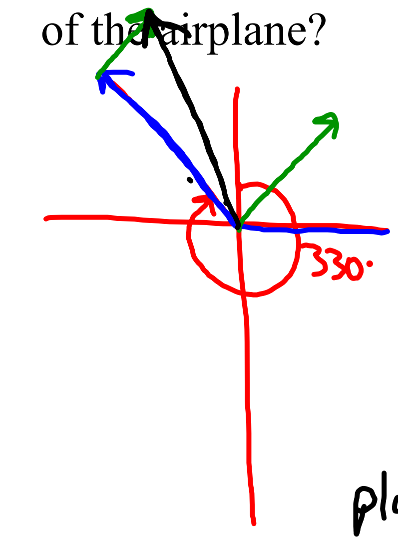


Using Vectors to Find Speed and Direction *Example 10 p.416*

° clockwise from N

$||v|| = 500$

An airplane is traveling at a speed of 500 mph with a bearing of 330°. The airplane encounters a wind blowing 70 mph in the direction N 45° E. What are the resultant speed and direction of the airplane?



plane:

$500\cos 120,\ 500\sin 120$

$500\left(-\frac{1}{2}\right),\ 500\frac{\sqrt{3}}{2}$

$\langle -250,\ 250\sqrt{3} \rangle$

wind: $\langle 35\sqrt{2},\ 35\sqrt{2} \rangle$

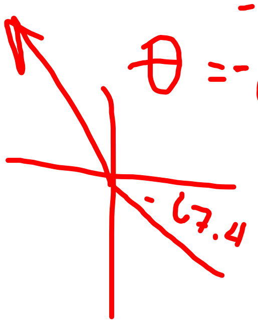
plane wind:

$\langle -200.5,\ 482.5 \rangle$

Speed $= ||v|| = \sqrt{(-200.5)^2 + (482.5)^2}$

$||v|| = 522.5$ mph

$\tan\theta = \frac{482.5}{-200.5}$



$\theta = -67.4 + 180 = \boxed{112.6°}$