6.4 Vectors and Dot Products

The Dot Product of Two Vectors

Why you should learn it

You can use the dot product of two vectors to solve real-life problems involving two vector quantities. For instance, Exercise 73 on page 441 shows you how the dot product can be used to find the force necessary to keep a truck from rolling down a hill.

Definition of Dot Product

The **dot product** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Example 1: Find the dot product: $\langle 5, -4 \rangle \cdot \langle 9, -2 \rangle$

45 + 8

53

b. <2, -1>•<1, 2>

2 + -2 = 0

c. <0, 3>•<4, -2>

0 + -6

-6

Properties of the Dot Product (See the proofs on page 467.)

Let **u**, **v**, and **w** be vectors in the plane or in space and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{0} \cdot \mathbf{v} = 0$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
5. $c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v}$

Properties of Dot Products

Let **u** = <-1, 3>, **v** = <2, -4>, and **w** = <1, -2>
Find the quantity. Is the result a vector or a scalar?

a. (u•v)w

-2 + -12

-14 <1, -2> = <-14, 28>

b. u•2v

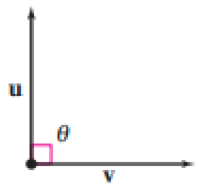
<-1, 3> · <4, -8>

-4 + -24

-28

Definition of Orthogonal Vectors

The vectors **u** and **v** are **orthogonal** when $\mathbf{u} \cdot \mathbf{v} = 0$.



$\theta = \frac{\pi}{2}$
$\cos\theta = 0$
90° angle

The term orthogonal and perpendicular mean essentially the same thing-meeting at right angles.

To find parallel you take $\frac{y}{x}$ from <x, y>

Two vectors **u** and **v** are orthogonal if $u \cdot v = 0$.

Two vectors **u** and **v** are parallel if y/x.

Are the vectors $\mathbf{u} = \langle 1, -4 \rangle$ and $\mathbf{v} = \langle 6, 2 \rangle$ orthogonal, parallel or neither?

$6 + -8$

-2

$\frac{-4}{1}$ $\frac{2}{6}$

$-4 = \frac{1}{3}$

Neither

Determine whether u and v are orthongonal, parallel, or neither.

1. u = <2, 5>
   v = <-15, 6>

2. u = <-15, 51>
   v = <20, -68>

3. u = <8, 5>
   v = <-2, 4>

Find the value of k such that the vectors u and v are orthogonal.

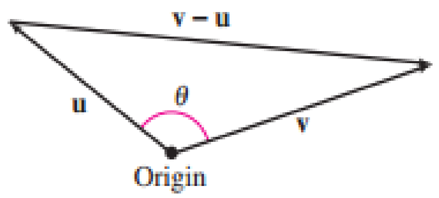
u = 2i + j
v = -i - kj

$2 \cdot -1 + -k = 0$

$-2 + -k = 0$

$k = -2$

The Angle Between Two Vectors



Angle Between Two Vectors (See the proof on page 467.)

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

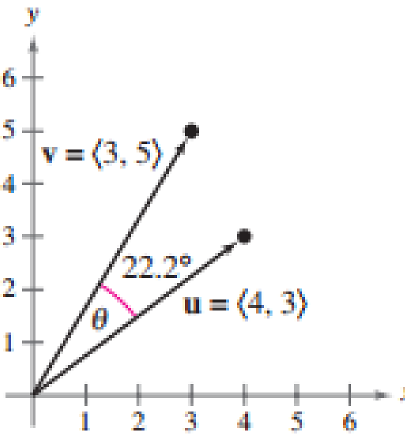
$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

$\mathbf{u} = \langle 4, 3 \rangle$ and $\mathbf{v} = \langle 3, 5 \rangle$.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

$$= \frac{\langle 4, 3 \rangle \cdot \langle 3, 5 \rangle}{\|\langle 4, 3 \rangle\|\,\|\langle 3, 5 \rangle\|}$$

$$= \frac{27}{5\sqrt{34}}$$



This implies that the angle between the two vectors is

$$\theta = \arccos\frac{27}{5\sqrt{34}} \approx 22.2°$$

Find the angle θ between the vectors.

$u \cdot v = 8$

u = 2i -3j
v= i - 2j

$\frac{8}{\sqrt{13} \cdot \sqrt{5}}$

$\|u\| = \sqrt{13}$

$\theta \approx 7.125°$

$\|v\| = \sqrt{5}$

Find the angle θ between the vectors.

$u = 3i - 5j$
$v = 5i + 6j$

The work done by a force **F** on a vector **v** is:

$$W = \left(\frac{F \bullet v}{\|v\|^2} \right) v$$

Turn to your right and explain how to do this problem to your neighbor.

Find the dot product:

$u = \langle 5, 7 \rangle \qquad v = \langle -2, 9 \rangle$

$u \bullet v$

Now have the other person explain how to do this problem to your neighbor.

Find the angle θ between the vectors.

u = 2i -3j

v= i - 2j

What does it mean to be orthogonal?

In the figure below, a = 12, b = 8 and and $\theta = 65^o$. Use the information to solve the parallelogram for c and d. The diagonals of the parallelogram are represented by c and d. Round your answer to 2 decimal places.

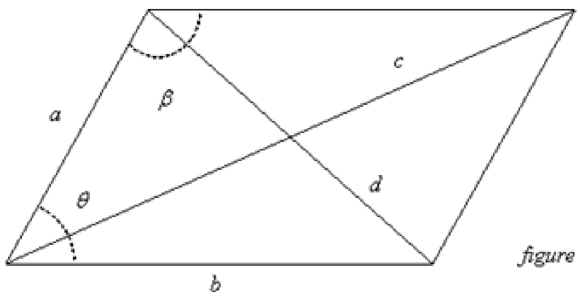
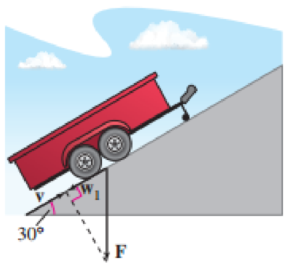


figure not drawn to scale

Finding a Force (Work)

A 200 lb go-cart sits on a ramp inclined at 30°. What force is required to keep the cart from rolling down the ramp?
(Force required = magnitude of the Work)



Solution

Because the force due to gravity is vertical and downward, you can represent the gravitational force by the vector

$\mathbf{F} = -200\mathbf{j}.$ Force due to gravity

To find the force required to keep the cart from rolling down the ramp, project **F** onto a unit vector **v** in the direction of the ramp, as follows.

$\mathbf{v} = (\cos 30°)\mathbf{i} + (\sin 30°)\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$ Unit vector along ramp

Therefore, the projection of **F** onto **v** is

$$\begin{aligned}\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{F} &= \left(\frac{\mathbf{F}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}\\ &= (\mathbf{F}\cdot\mathbf{v})\mathbf{v}\\ &= (-200)\left(\frac{1}{2}\right)\mathbf{v}\\ &= -100\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right).\end{aligned}$$

The magnitude of this force is 100, and therefore a force of 100 pounds is required to keep the cart from rolling down the ramp.