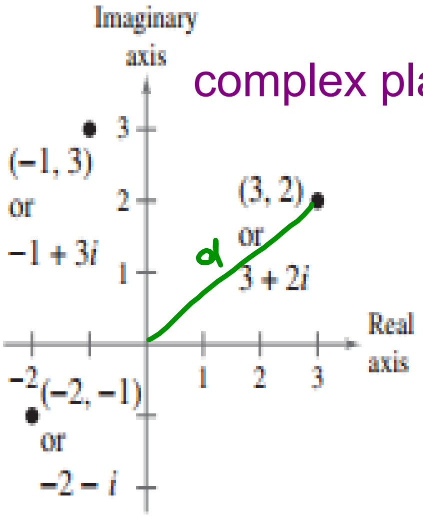
# 6.5 Trigonometric Form of a Complex Number




The absolute value of a complex number a + bi is defined as the distance between the origin (0,0) and the point (a, b).

**Definition of the Absolute Value of a Complex Number**

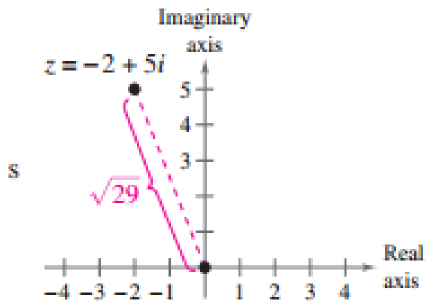
The **absolute value** of the complex number $z = a + bi$ is given by

$$|a + bi| = \sqrt{a^2 + b^2}.$$

## Finding the Absolute Value of a Complex Number

z = -2 + 5i

$$|z| = \sqrt{(-2)^2 + 5^2}$$
$$= \sqrt{29}.$$



$\cos\theta = \frac{a}{r}$

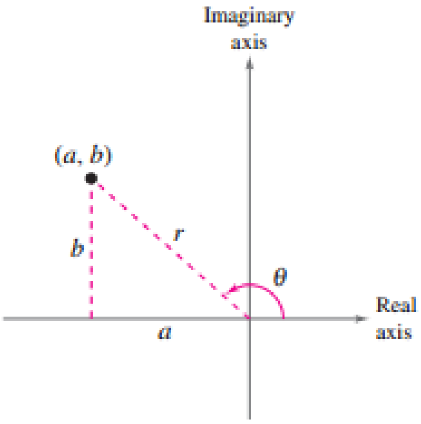
$\sin\theta = \frac{b}{r}$

## Trigonometric Form of a Complex Number

$a = r\cos\theta$ and $b = r\sin\theta$

where

$$r = \sqrt{a^2 + b^2}$$




Consequently, you have

$$a + bi = (r\cos\theta) + (r\sin\theta)i$$

### Trigonometric Form of a Complex Number

The **trigonometric form** of the complex number $z = a + bi$ is given by

$$z = r(\cos\theta + i\sin\theta)$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = b/a$. The number $r$ is the **modulus** of $z$, and $\theta$ is called an **argument** of $z$.

## Example 2 Writing a Complex Number in Trigonometric Form

Write the complex number

$z = -2i$

in trigonometric form.

Figure out r

**Solution**

The absolute value of $z$ is

$$r = |-2i| = \sqrt{0^2 + (-2)^2} = \sqrt{4} = 2.$$

With $a = 0$, you cannot use $\tan \theta = b/a$ to find $\theta$. Because $z = -2i$ lies on the negative imaginary axis (see Figure 6.50), choose $\theta = 3\pi/2$. So, the trigonometric form is

$$z = r(\cos \theta + i \sin \theta)$$

$$= 2\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right).$$

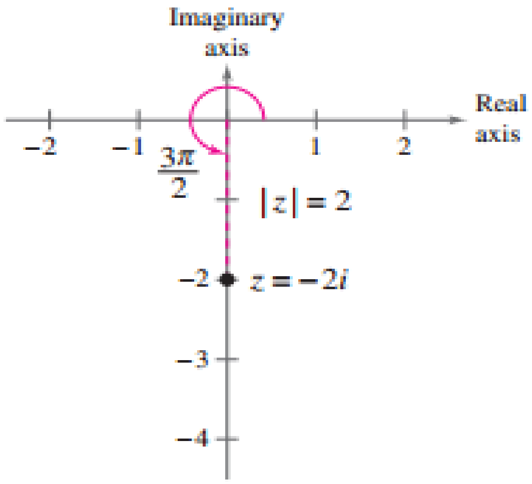



Figure 6.50

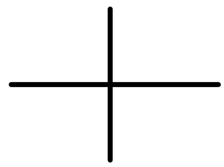
## Example 3 Writing a Complex Number in Trigonometric Form

Write the complex number $z = -2 - 2\sqrt{3}i$ in trigonometric form.

### Solution

$a = -2 \quad b = -2\sqrt{3}$

sketch a graph



The absolute value of $z$ is

$$r = \left|-2 - 2\sqrt{3}i\right| = \sqrt{(-2)^2 + \left(-2\sqrt{3}\right)^2} = \sqrt{16} = 4$$

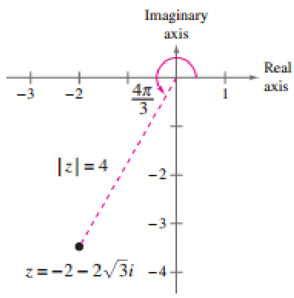
and the angle $\theta$ is given by

$$\tan\theta = \frac{b}{a} = \frac{-2\sqrt{3}}{-2} = \sqrt{3}.$$

Because $\tan(\pi/3) = \sqrt{3}$ and $z = -2 - 2\sqrt{3}i$ lies in Quadrant III, choose $\theta$ to be $\theta = \pi + \pi/3 = 4\pi/3$. So, the trigonometric form is

$$z = r(\cos\theta + i\sin\theta)$$

$$= 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right).$$



## Example 4 Writing a Complex Number in Standard Form

Write the complex number in standard form $a + bi$.

$$z = \sqrt{8}\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$$

### Solution

Because $\cos(-\pi/3) = 1/2$ and $\sin(-\pi/3) = -\sqrt{3}/2$, you can write

$$\begin{aligned} z &= \sqrt{8}\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right] \\ &= \sqrt{8}\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right] \\ &= 2\sqrt{2}\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right] \\ &= \sqrt{2} - \sqrt{6}i. \end{aligned}$$

### Product and Quotient of Two Complex Numbers

Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ be complex numbers.

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$ Product

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)], \quad z_2 \neq 0$$ Quotient

## Example 5 Multiplying Complex Numbers in Trigonometric Form

Find the product $z_1z_2$ of the complex numbers.

$$z_1 = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$z_2 = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$z_1z_2 = r_1r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

**Solution**

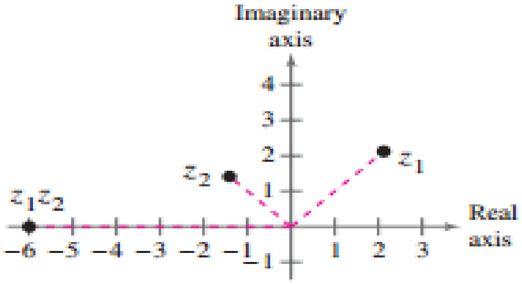
$$z_1z_2 = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\cdot 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= 6\left[\cos\left(\frac{\pi}{4}+\frac{3\pi}{4}\right) + i\sin\left(\frac{\pi}{4}+\frac{3\pi}{4}\right)\right]$$

$$= 6(\cos\pi + i\sin\pi)$$

$$= 6[-1 + i(0)]$$

$$= -6$$




The numbers $z_1$, $z_2$, and $z_1z_2$ are plotted

## Example 7 Dividing Complex Numbers in Trigonometric Form

np

Find the quotient

$$\frac{z_1}{z_2}$$

of the complex numbers.

$z_1 = 24(\cos 300° + i \sin 300°)$ $z_2 = 8(\cos 75° + i \sin 75°)$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + \sin\left(\theta_1 - \theta_2\right)\right]$$

$$\frac{z_1}{z_2} = \frac{24(\cos 300° + i \sin 300°)}{8(\cos 75° + i \sin 75°)}$$

$$= \frac{24}{8}[\cos(300° - 75°) + i \sin(300° - 75°)]$$

$$= 3(\cos 225° + i \sin 225°)$$

$$= 3\left[\left(-\frac{\sqrt{2}}{2}\right) + i\left(-\frac{\sqrt{2}}{2}\right)\right]$$

$$= -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$$

## Powers of Complex Numbers

**DeMoivre's Theorem**

If $z = r(\cos\theta + i\sin\theta)$ is a complex number and $n$ is a positive integer, then

$$z^n = [r(\cos\theta + i\sin\theta)]^n$$
$$= r^n(\cos n\theta + i\sin n\theta).$$

**DeMoivre's Theorem**

If $z = r(\cos\theta + i\sin\theta)$ is a complex number and $n$ is a positive integer, then

$$\begin{aligned} z^n &= [r(\cos\theta + i\sin\theta)]^n \\ &= r^n(\cos n\theta + i\sin n\theta). \end{aligned}$$

## Example 8 Finding a Power of a Complex Number

Use DeMoivre's Theorem to find

$$\left(1 + \sqrt{3}i\right)^{12}.$$

First convert the complex number to trigonometric form using

$$r = \sqrt{(1)^2 + \left(\sqrt{3}\right)^2} = 2$$

and

$$\theta = \arctan\frac{\sqrt{3}}{1} = \frac{\pi}{3}.$$

So, the trigonometric form is

$$1 + \sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).$$

Then, by DeMoivre's Theorem, you have

$$\begin{aligned} \left(1 + \sqrt{3}i\right)^{12} &= \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^{12} \\ &= 2^{12}\left(\cos\frac{12\pi}{3} + i\sin\frac{12\pi}{3}\right) \\ &= 4096(\cos 4\pi + i\sin 4\pi) \\ &= 4096(1 + 0) \\ &= 4096. \end{aligned}$$

## Partner Practice

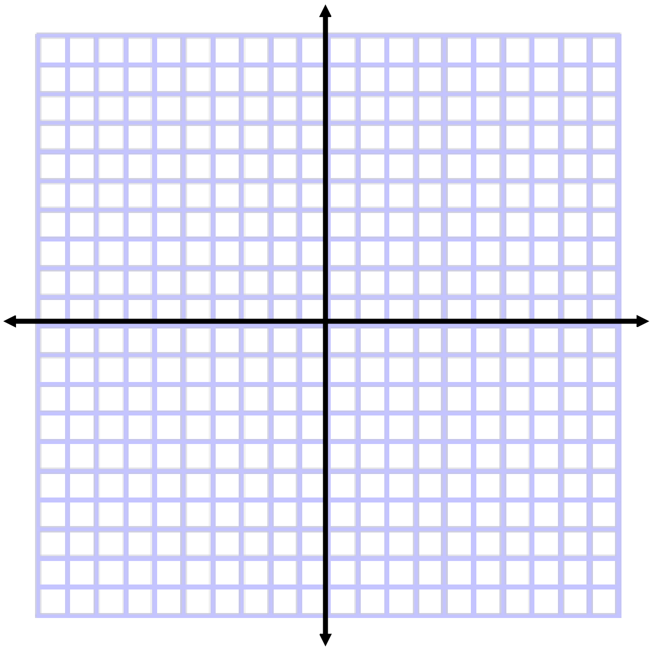
$$(2 + 2i)(1 - i)$$

a. Write trigonometric forms of the complex numbers
b. Perform the operation using trig form
c. Perform the operation using standard form
d. Check your answers

## Finding the Absolute Value of a Complex Number

$z = 5 + 7i$

$z = -3 - 4i$



Write the complex number of $z = 6 - 6i$ in trigonometric form.

Write the complex number $z = 3i$ in trigonometric form.

Write the complex number in standard form a + bi

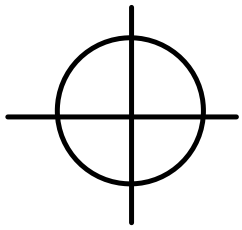
$$z = 8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

## Example 6 Multiplying Complex Numbers in Trigonometric Form

Find the product $z_1 z_2$ of the complex numbers.

$$z_1 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \qquad z_2 = 8\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

$$
\begin{aligned}
z_1 z_2 &= 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \cdot 8\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) \\
&= 16\left[\cos\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right) + i\sin\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right)\right] \\
&= 16\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) \\
&= 16\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \\
&= 16[0 + i(1)] \\
&= 16i
\end{aligned}
$$



You can check this result by first converting to the standard forms

$$z_1 = -1 + \sqrt{3}i \quad \text{and} \quad z_2 = 4\sqrt{3} - 4i$$

and then multiplying algebraically, as in Section 2.4.

$$
\begin{aligned}
z_1 z_2 &= (-1 + \sqrt{3}i)(4\sqrt{3} - 4i) \\
&= -4\sqrt{3} + 4i + 12i + 4\sqrt{3} \\
&= 16i
\end{aligned}
$$

Use DeMoivre's Theorem to find $(1+i)^6$