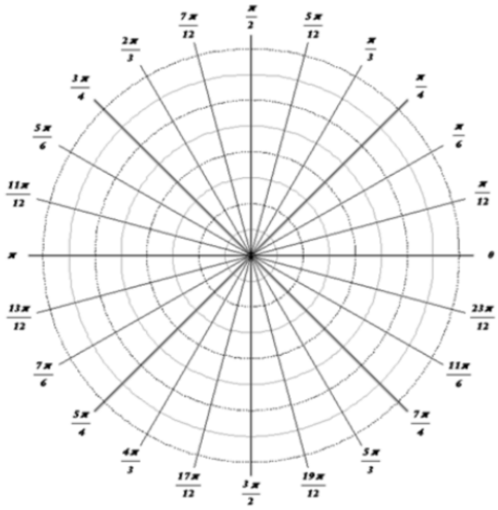
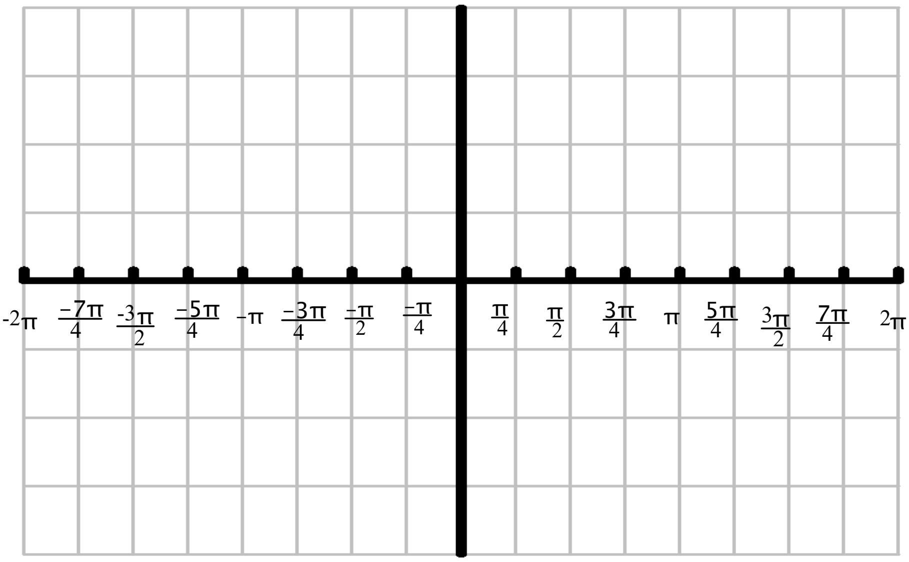
9.6 Graphs of Polar Equations

r = 4


π/2
7π/12
5π/12
2π/3
π/3
3π/4
π/4
5π/6
π/6
11π/12
π/12
π
0
13π/12
23π/12
7π/6
11π/6
5π/4
7π/4
4π/3
5π/3
17π/12
19π/12
3π/2

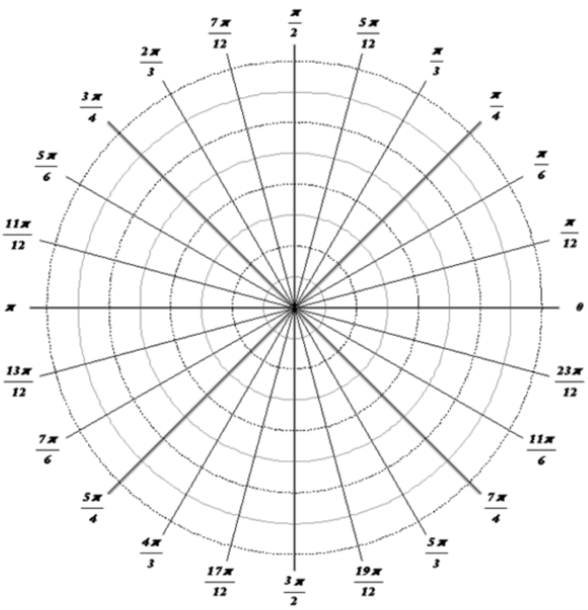
Graph it in rectangular first

$r = 4 \sin \theta$



Complete Chart

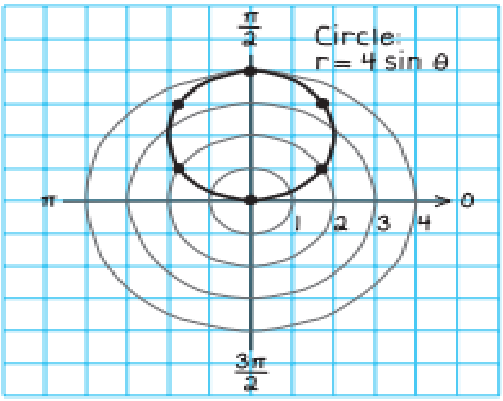
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
r	0	2	$2\sqrt{3}$	4	$2\sqrt{3}$	2	0	-2	-4	-2	0



Example 1 Graphing a Polar Equation by Point Plotting

Sketch the graph of the polar equation $r = 4 \sin \theta$ by hand.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
r	0	2	$2\sqrt{3}$	4	$2\sqrt{3}$	2	0	-2	-4	-2	0



You can confirm the graph found in Example 1 in three ways.

1. *Convert to Rectangular Form* Multiply each side of the polar equation by r and convert the result to rectangular form.
2. *Use a Polar Coordinate Mode* Set your graphing utility to *polar* mode and graph the polar equation. (Use $0 \le \theta \le \pi$, $-6 \le x \le 6$, and $-4 \le y \le 4$.)
3. *Use a Parametric Mode* Set your graphing utility to *parametric* mode and graph $x = (4 \sin t) \cos t$ and $y = (4 \sin t) \sin t$.

Most graphing utilities have a *polar* graphing mode. If yours doesn't, you can rewrite the polar equation $r = f(\theta)$ in parametric form, using t as a parameter, as follows.

$$x = f(t) \cos t \qquad \text{and} \qquad y = f(t) \sin t$$

Testing for Symmetry in Polar Coordinates

The graph of a polar equation is symmetric with respect to the following when the given substitution yields an equivalent equation.

1. The line $\theta = \frac{\pi}{2}$: Replace (r, θ) by $(r, \pi - \theta)$ or $(-r, -\theta)$.
2. The polar axis: Replace (r, θ) by $(r, -\theta)$ or $(-r, \pi - \theta)$.
3. The pole: Replace (r, θ) by $(r, \pi + \theta)$ or $(-r, \theta)$.

You can determine the symmetry of the graph of $r = 4 \sin \theta$ (see Example 1) as follows.

1. Replace (r, θ) by $(-r, -\theta)$:

$$-r = 4\sin(-\theta) \quad \Rightarrow \quad r = -4\sin(-\theta) = 4\sin\theta$$

2. Replace (r, θ) by $(r, -\theta)$:

$$r = 4\sin(-\theta) = -4\sin\theta$$

3. Replace (r, θ) by $(-r, \theta)$:

$$-r = 4\sin\theta \quad \Rightarrow \quad r = -4\sin\theta$$

So, the graph of $r = 4 \sin \theta$ is symmetric with respect to the line $\theta = \pi/2$.

Example 2 Using Symmetry to Sketch a Polar Graph

Use symmetry to sketch the graph of

$$r = 3 + 2\cos\theta$$

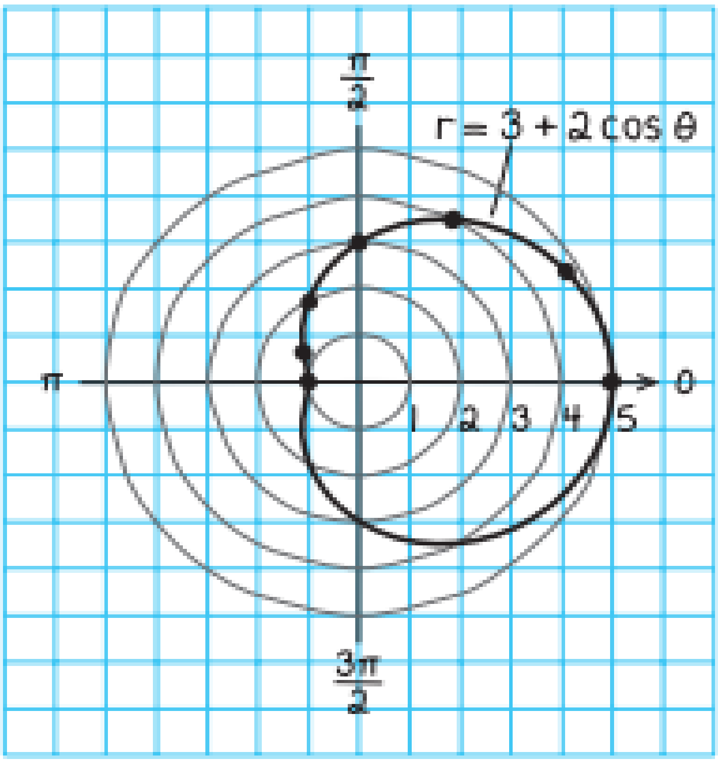
Solution

Replacing (r, θ) by $(r, -\theta)$ produces

$$r = 3 + 2\cos(-\theta) = 3 + 2\cos\theta. \qquad \cos(-u) = \cos u$$

So, by using the even trigonometric identity, you can conclude that the curve is symmetric with respect to the polar axis. Plotting the points in the table and using polar axis symmetry, you obtain the graph shown in Figure 9.72. This graph is called a **limaçon.**

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	5	$3+\sqrt{3}$	4	3	2	$3-\sqrt{3}$	1



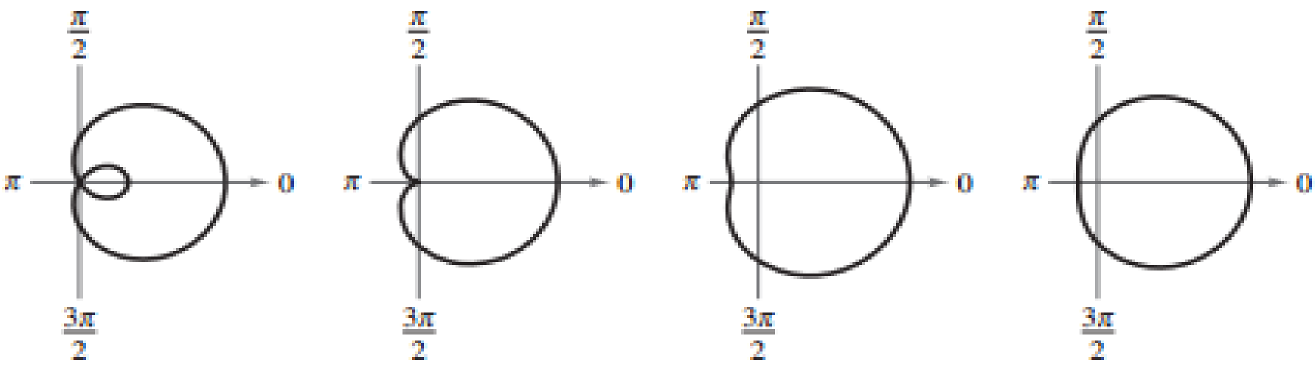
Quick Tests for Symmetry in Polar Coordinates

1. The graph of $r = f(\sin \theta)$ is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.

2. The graph of $r = g(\cos \theta)$ is symmetric with respect to the polar axis.

Limaçons

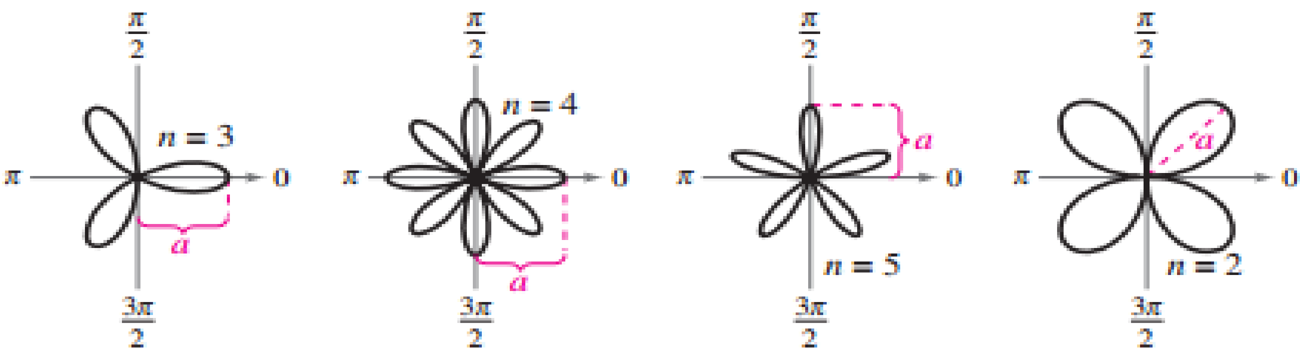
$r = a \pm b\cos\theta,\ r = a \pm b\sin\theta \quad (a > 0, b > 0)$



$\frac{a}{b} < 1$	$\frac{a}{b} = 1$	$1 < \frac{a}{b} < 2$	$\frac{a}{b} \geq 2$
Limaçon with inner loop	Cardioid (heart-shaped)	Dimpled limaçon	Convex limaçon

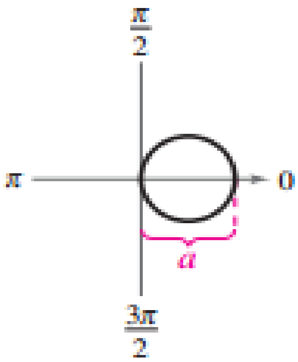
Rose Curves

n petals when n is odd, $2n$ petals when n is even $(n \geq 2)$

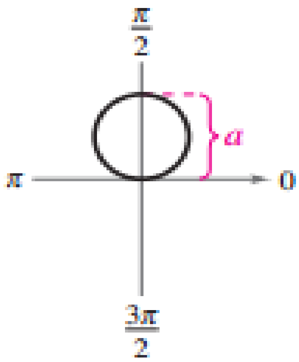


$r = a \cos n\theta$ Rose curve | $r = a \cos n\theta$ Rose curve | $r = a \sin n\theta$ Rose curve | $r = a \sin n\theta$ Rose curve

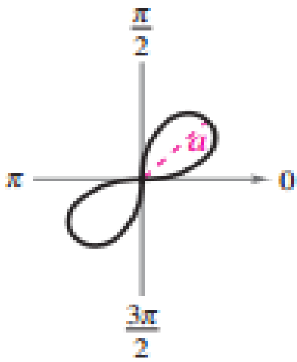
Circles and Lemniscates



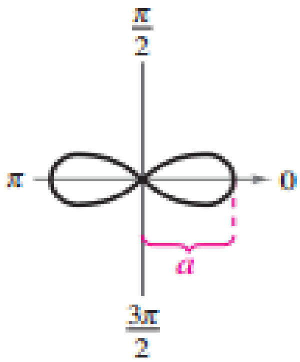
$r = a \cos \theta$
Circle



$r = a \sin \theta$
Circle



$r^2 = a^2 \sin 2\theta$
Lemniscate



$r^2 = a^2 \cos 2\theta$
Lemniscate

Example 3 Analyzing a Polar Graph

Analyze the graph of

$$r = 2 \cos 3\theta.$$

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	$-\sqrt{2}$	0

Example 4 Analyzing a Rose Curve

Analyze the graph of $r = 3\cos 2\theta$.

Solution

Type of curve: Rose curve with $2n = 4$ petals

Symmetry: With respect to the polar axis, the line $\theta = \frac{\pi}{2}$, and the pole

Zeros of r: $r = 0$ when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

Using a graphing utility, enter the equation, as shown in Figure 9.75 (with $0 \leq \theta \leq 2\pi$). You should obtain the graph shown in Figure 9.76.

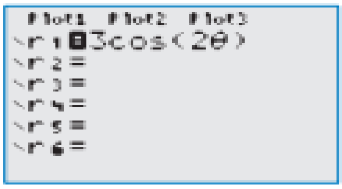


Figure 9.75

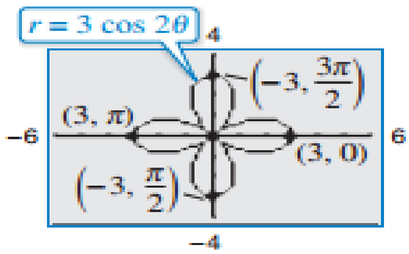


Figure 9.76

Example 5 Analyzing a Lemniscate

Analyze the graph of $r^2 = 9 \sin 2\theta$.

Solution

Type of curve: Lemniscate

Symmetry: With respect to the pole

Zeros of r: $r = 0$ when $\theta = 0, \frac{\pi}{2}$

Using a graphing utility, enter the equation, as shown in Figure 9.77 (with $0 \le \theta \le 2\pi$). You should obtain the graph shown in Figure 9.78.

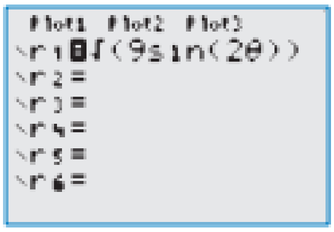


Figure 9.77

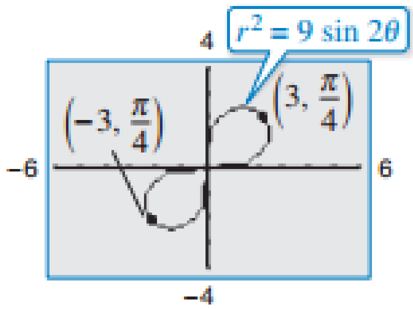


Figure 9.78

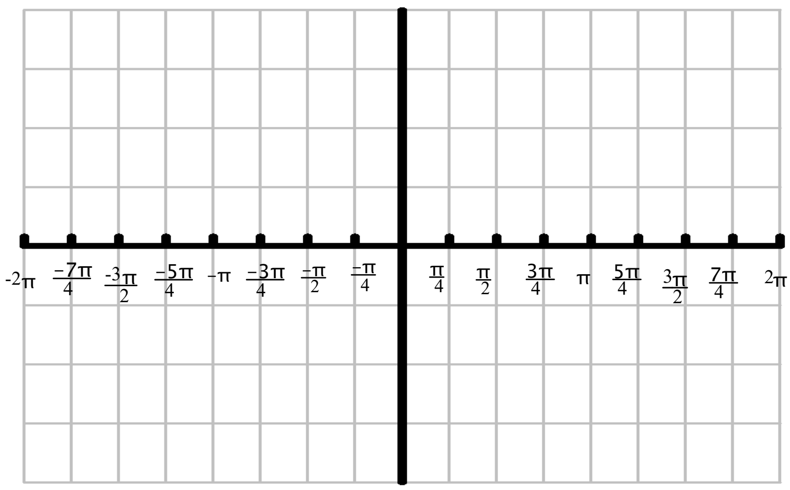
Graph the following with a partner name each one! NO Calculators!

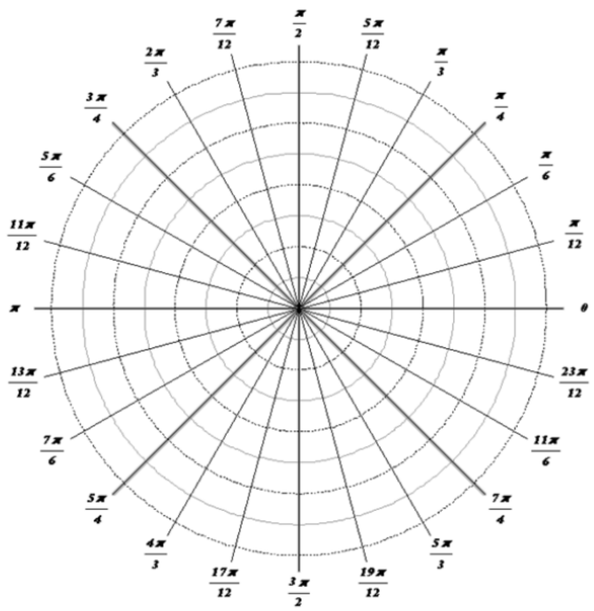
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
r											

$r = 5 \cos 3\theta$

$r = 2 + 4\cos \theta$

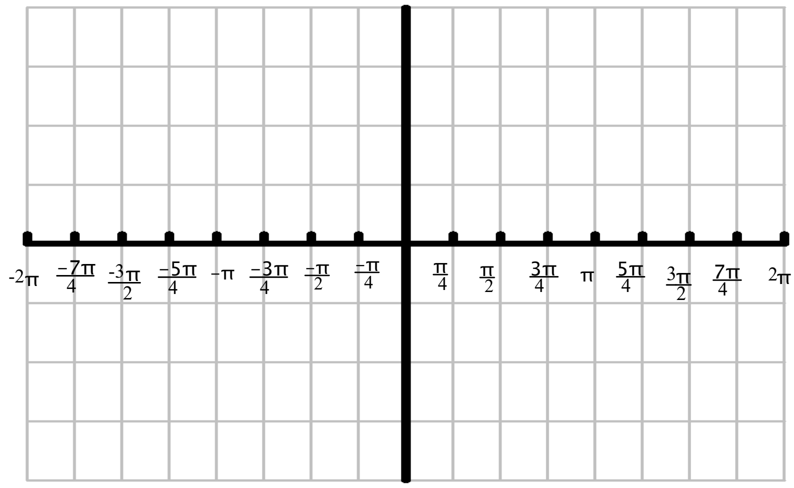
$r = 1 - 2 \sin \theta$

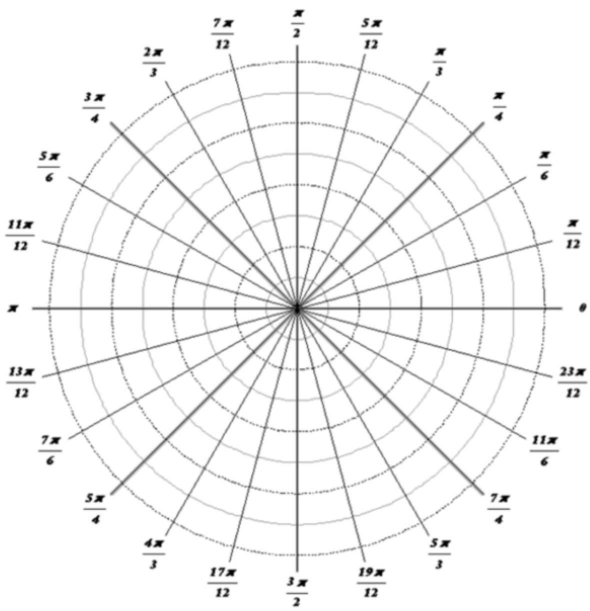




$r = 2 + 4\cos\theta$

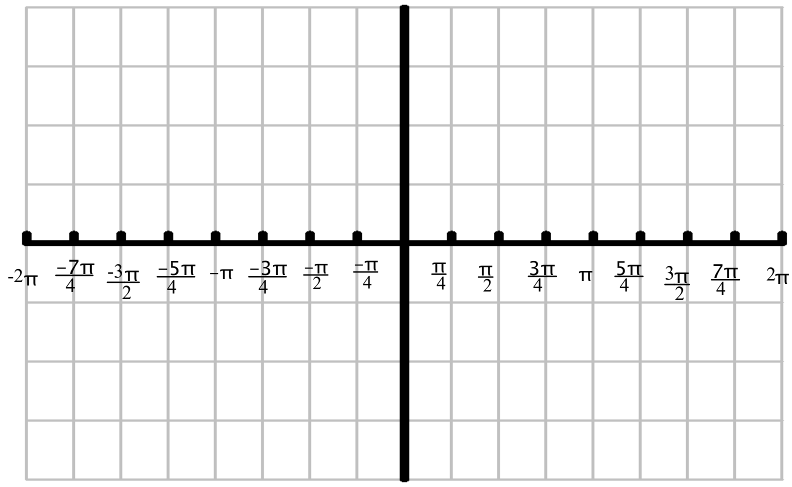
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
r											

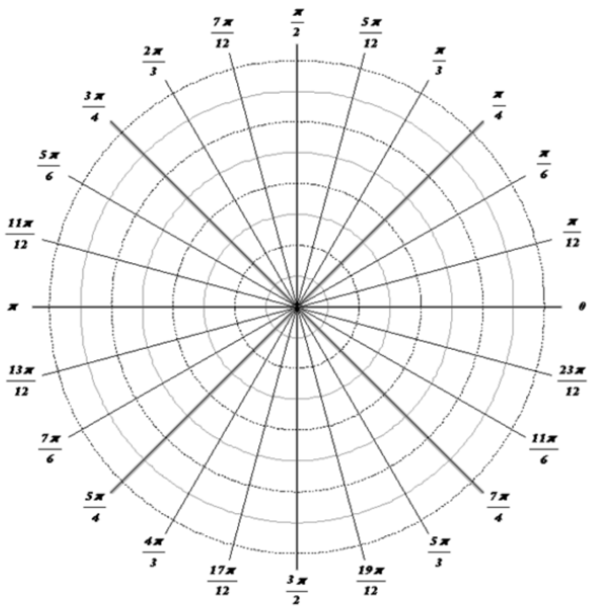




$r = 1 - 2\sin\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
r											





Graph these 2 using either chart or graph, your choice with a partner. Finish your worksheet and work on homework the remaining period

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
r											

$$r^2 = 9\sin 2\theta$$
$$r = 2\sin 3\theta$$

Name it!